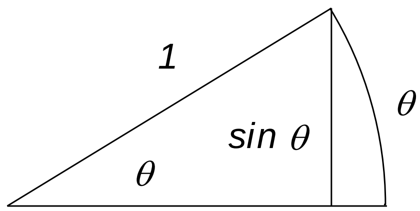
Math 109 Calc 1 Lecture 8

Derivatives of the Trigonometric (Circular) Functions

Derivative of sin(x)

To find [sin(x)]' we're going to need two special limits.

First, we need a simple result, that for small $\theta, sin(\theta) \leq \theta$



Looking at this diagram the arc of the sector clearly must be less than or equal to the vertical line segment showing the result we want.

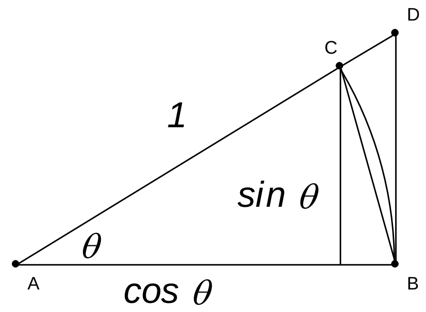
Next, we want to find that $\lim_{h \to 0} \frac{1-cos(h)}{h} = 0$

Since $sin(h) \leq h$ we know that $\frac{1-cos(h)}{h} \leq \frac{1-cos(h)}{sin(h)}$ because on the right side we are dividing by a smaller number.

We can transform $\frac{1-cos(h)}{sin(h)} \cdot \frac{1+cos(h)}{1+cos(h)} = \frac{1-cos^2(h)}{sin(h)(1+cos(h))} = \frac{sin^2(h)}{sin(h)(1+cos(h))} = \frac{sin(h)}{(1+cos(h))}$

So we have that $\lim_{h \to 0} \frac{1-cos(h)}{h} \leq \lim_{h \to 0} \frac{1-cos(h)}{sin(h)} = \lim_{h \to 0} \frac{sin(h)}{(1+cos(h))} = \frac{0}{1} = 0$

Next, we want to show that $\lim_{h \to 0} \frac{\sin(h)}{h} = 1$



A few observations here.
Since the area of a unit circle is $\pi r^2 = \pi$
and the ratio of the area of the sector shown to the area of a circle is

$\frac{A}{\pi} = \frac{\theta}{2\pi}$ we can conclude that $A = \frac{\theta\pi}{2\pi} = \frac{\theta}{2}$

This area is clearly greater than the area of the smaller triangle ABC.
The area of this triangle is $\frac{1}{2}\sin(\theta)$
It's also less than the area of the larger triangle ABD.

By the similarity of triangles ABC and ABD we have that

$\frac{BD}{1} = \frac{\sin(\theta)}{\cos(\theta)}$ so, $BD = \frac{\sin(\theta)}{\cos(\theta)}$

That being the case, triangle ABD has area $\frac{1}{2}\frac{\sin(\theta)}{\cos(\theta)}$

Comparing the areas, we have $\frac{1}{2} sin(\theta) \leq \frac{\theta}{2} \leq \frac{1}{2}\frac{sin(\theta)}{cos(\theta)}$

Multiplying by two and finding the reciprocals we get

$$\frac{1}{\sin(\theta)} \geq \frac{1}{\theta} \geq \frac{\cos(\theta)}{\sin(\theta)}$$

Multiplying by $\sin(\theta)$ and switching the order we have

$$cos(\theta) \leq \frac{sin(\theta)}{\theta} \leq 1$$

Now using the pinching theorem we can see that since

$\lim_{h \to 0} cos(h) = 1$ that $\lim_{h \to 0} \frac{sin(h)}{h} = 1$

Back to the derivative of the sine function

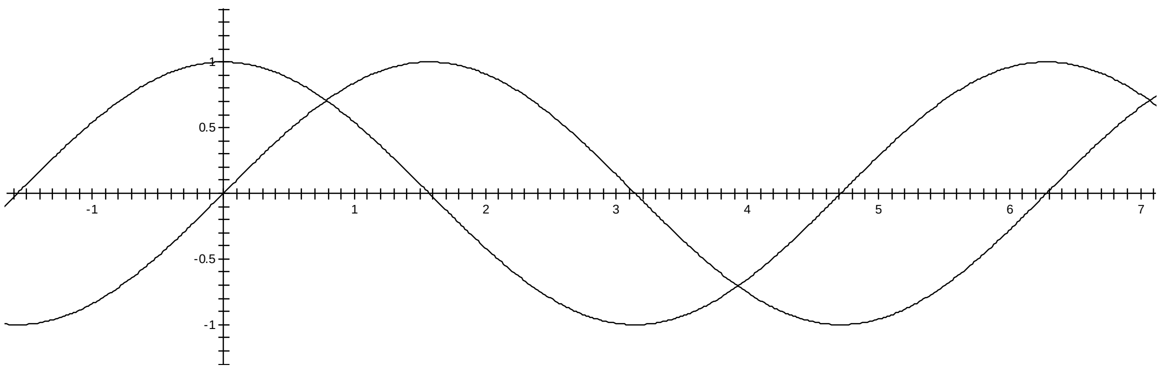
$$[sin(x)]' = \lim_{h\to 0} \frac{sin(x+h) - sin(x)}{h} = \lim_{h\to 0} \frac{sin(x)cos(h) + cos(x)sin(h) - sin(x)}{h} =$$

$$\lim_{h\to 0} \frac{sin(x)cos(h) - sin(x)}{h} + \lim_{h\to 0} \frac{cos(x)sin(h)}{h} =$$

$$-sin(x) \lim_{h\to 0} \frac{(1 - cos(h))}{h} + cos(x)\lim_{h\to 0} \frac{sin(h)}{h} = -sin(x)(0) + cos(x)(1) = cos(x)$$

So, we have the result that $[sin(x)]'$ = cos(x)

Using the same method, we can find that $[\cos(x)]'$ = -sin(x)



Let's look at this graph of sin(x) and cos(x). At zero the sine function appears to have a slope of 1 where cos(*x*)=1. Then going right at $\frac{\pi}{2}$ where sin(*x*) flattens out and has a slope of zero, we see that cos(*x*) = 0.

Example:

$[x^2\sin(x)]' = x^2\cos(x) + 2x\sin(x)$

Other Derivatives

Using the quotient rule we find that

$$[\tan(\theta)]' = \left[\frac{\sin(\theta)}{\cos(\theta)}\right]' = \frac{\cos(x)\cos(x)\text{-}(\text{-}\sin(x))\sin(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} = \sec^2(x)$$

Using the quotient and reciprocal rules gives us the other 3 derivatives.

$[\text{ctn}(\theta)]' = -\csc^2(\theta)$

$[\sec(\theta)]' = \sec(\theta)\tan(\theta)$

$[\csc(\theta)]' = -\csc(\theta)\text{ctn}(\theta)$