Math 109 Calc 1 Lecture 5

Derivatives + Rate of Change

Why Derivatives

We are going to spend a lot of time over the next few classes learning about derivatives and how to differentiate functions.

It might seem like a lot of effort for an operation that might have limited use.

So, before we get into the details, I'd like to give you an overview about why this operation is so useful.

First, I'm going to start with a very standard view.

Driving down the highway

Say that you are out driving down a long fast highway. There's a highway that you can take between San Francisco and Los Angeles called I 5 or Interstate 5. Much of this highway is straight and not all that scenic. You can fly down I 5 at from 70-80 without getting the attention of the state police. Go down Route 101 or even Route 1 if you want scenery. I 5 is just for getting to LA fast.

So, as you drive you have three gauges you can look at.

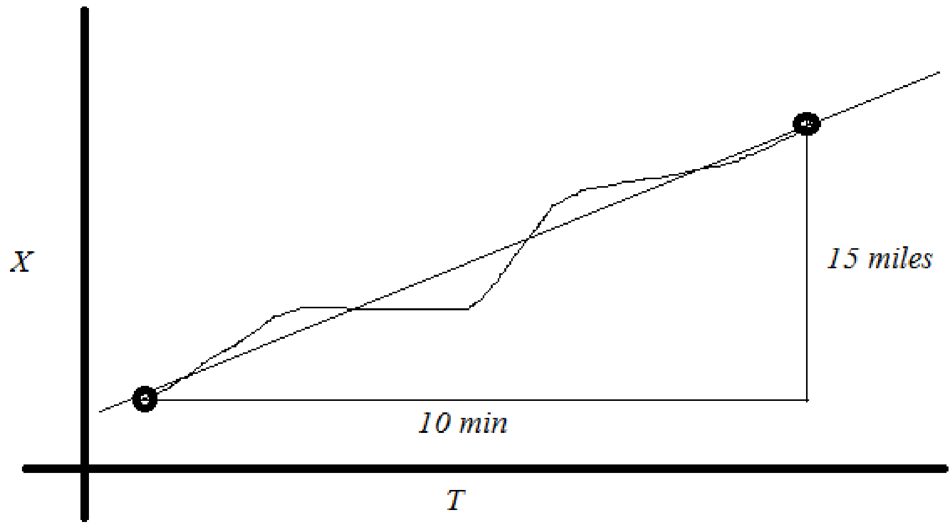
1. A clock – this lets you watch the minutes and hours go by, let's call this T
2. An odometer – this lets you watch the miles go by, 1/10 at a time, let's call this X
3. A speedometer - this tells you how fast you are going, let's call this V

Since you are bored, you keep track of the time and the odometer. You notice that 10 minutes (or 1/6 of an hour) goes by and you travel 15 miles. You realize this gives you enough information to calculate an average rate of travel.

$$\mathrm{V_A} = \frac{15}{1/6} = 80\mathrm{mph}$$

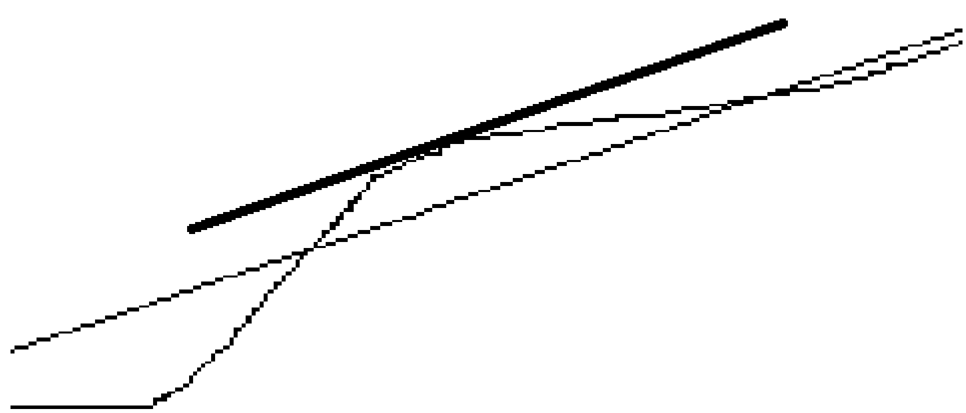
At the same time you watch the speedometer move back and forth between 75mph and 85mph.

Let's say you have a special computer in your car and at a rest stop you pull up a graph for this period of travel.



So here the slope of the straight line represents the velocity you would have been traveling if you drove at a constant speed. But clearly this is just an approximation. At any given moment, you speed will vary.

If you were to look at a specific place on the graph and draw a tangent to the curve at a point, the slope of that line would represent the velocity at that particular moment. This is what the speedometer is showing you.



Both of these values, **Average Velocity** and **Instantaneous Velocity** have their uses.

Average velocity is valuable if you want to estimate how long the rest of your trip will take.

Instantaneous velocity is useful for avoiding a speeding ticket.

Differentiation

The process of finding a curve whose value at each point in time represents slope of the tangent at that point is called **differentiation or** finding the **derivative** function.

This process may be done analytically if we have an explicit description of the function, or it could be done approximately using data points that are sampled and recorded. Over the next few days, we are going to learn how to differentiate a function analytically.

Rate of Change

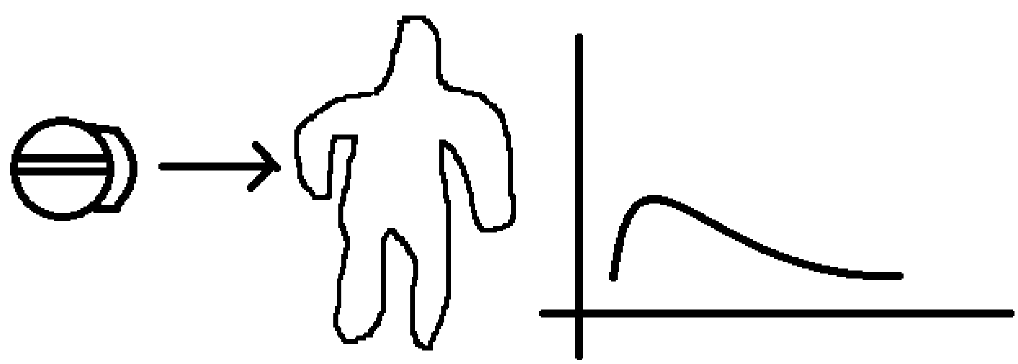
If all this was good for was finding velocity it would have limited use.

However, what we are finding here is a rate of change.

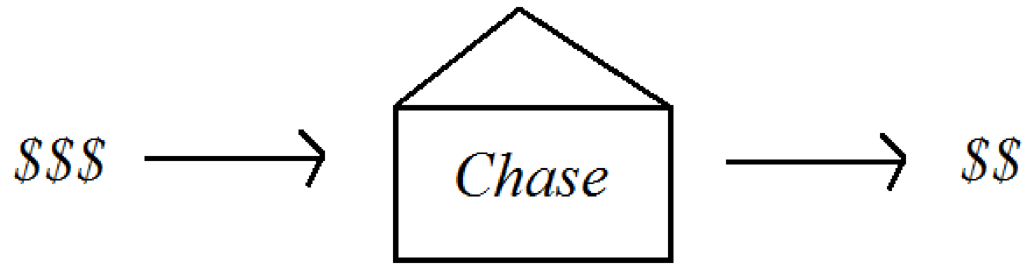
Many rates of change that we will look at will have time as the independent variable.

Here are some examples from the real world that you might want to consider.

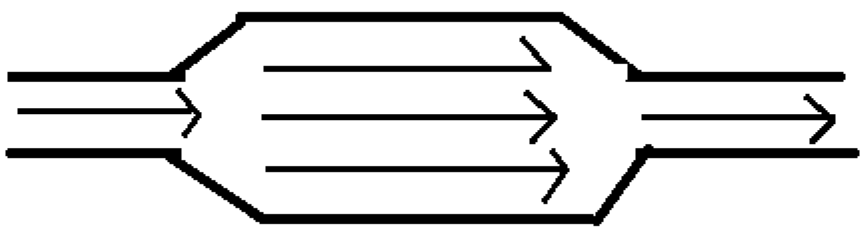
In medicine, a doctor may prescribe a drug for a patient. The rate that the drug is absorbed and removed from a human body will depend on several variables. One of the variables is the size and weight of the patient. Another important variable is the current concentration of the drug in the patient at a moment in time.



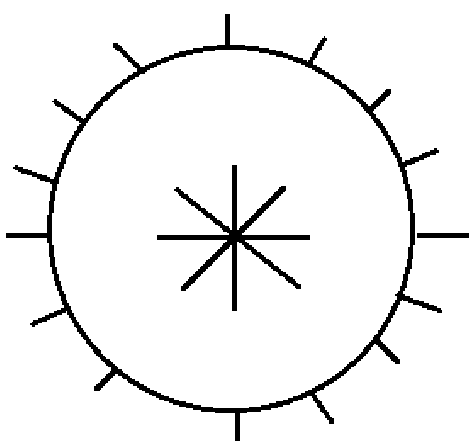
From Economics, money travels through the economy at different rates. Some of the variables that it will depend on are the amount of currency in the system, and the lending rate that banks are charging.



From Physics, the rate that a fluid such as water travels through a pipe will depend on the size of the pipe and the pressure.



From Astronomy, the rate at which energy travels from the core of a star out to the surface depends on the temperature, pressure and density of the matter

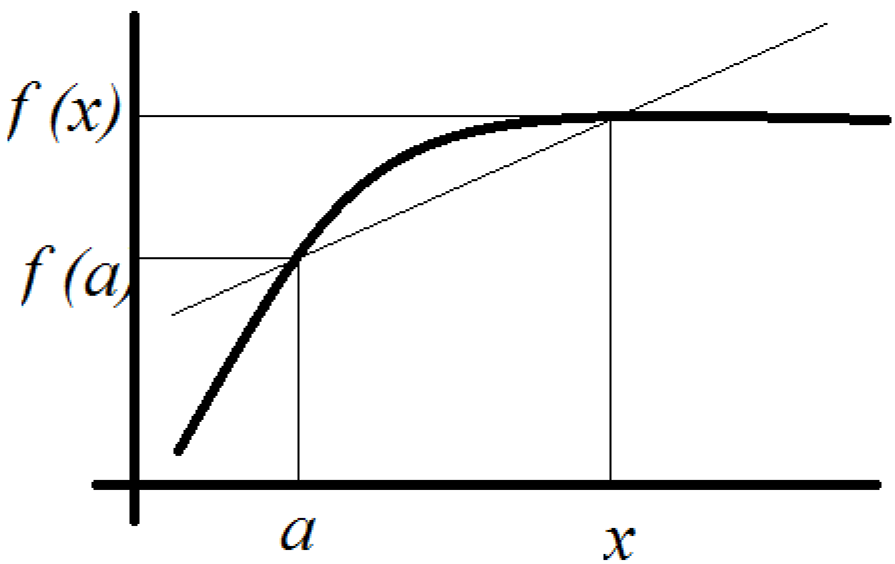


The Derivative at a Point

With this in mind, let's focus on how we find the derivative of a function by first looking at the slope at a particular point of a function.

We can approximate the slope by taking a nearby point and using the slope equation

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - xy_1}$$



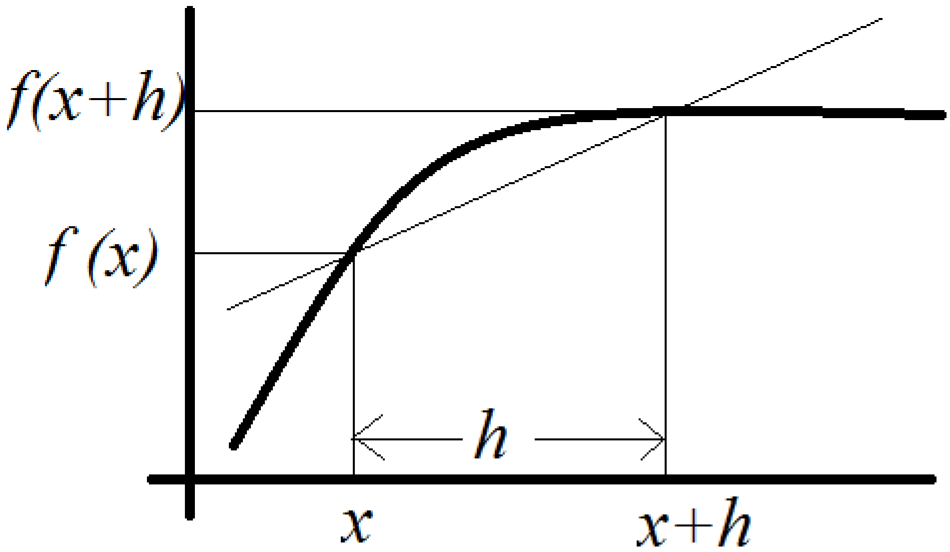
This gives us a formula for the approximate slope at *a*.

$$f'_{approx}(a) = \frac{f(x)-f(a)}{x-a}$$

In order to find the instantaneous rate at a, we let x approx. *a* and we find the limit

$$f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$$

This important formula can be written in a more useful way by substituting the value *h* = *x-a*



$$f'(x) = \lim_{h\to 0} \frac{f(x + h)-f(x)}{h}$$

This gives us a way to analytically find a derivative if we can find the limit.

Note that since *x* is unspecified here, this formula will give us the derivative as a function whose parameter is *x*.

Definition of a Derivative

A function *f* is said to be differentiable at *x* iff $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ exists.

Example: A parabola

$f(x) = x^2$

By the definition

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} =$$

$$\lim_{h \to 0} \frac{2xh + h^2}{h} = \lim_{h \to 0} \left(\frac{h}{h}\right)(2x + h) = 2x$$

Example: A line

$f(x) = mx + b$

By the definition

$$f'(x) = \lim_{h \to 0} \frac{m(x+h) + b - (mx + b)}{h} =$$

$$\lim_{h \to 0} \frac{mh}{h} = \lim_{h \to 0} \left(\frac{h}{h}\right) m = m$$

Which is of course what we expected of a straight line.

Example: A square root

$f(x) = \sqrt{x}$

By the definition

$$f'(x) = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \left(\frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}\right) =$$

$$\lim_{h \to 0} \frac{x + h - x}{h\left(\sqrt{x+h} + \sqrt{x}\right)} = \lim_{h \to 0} \left(\frac{h}{h}\right) \frac{1}{\left(\sqrt{x+h} + \sqrt{x}\right)} = \frac{1}{2\sqrt{x}}$$

Example: A repeat of a tangent to a circle

$f(x) = \sqrt{1-x^2}$

By the definition of a derivative

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{1-(x+h)^2}-\sqrt{1-x^2}}{h}\left(\frac{\sqrt{1-(x+h)^2}+\sqrt{1-x^2}}{\sqrt{1-(x+h)^2}+\sqrt{1-x^2}}\right) =$$

$$\lim_{h\to 0} \frac{1-(x+h)^2-(1-x^2)}{h\sqrt{1-(x+h)^2}+\sqrt{1-x^2}} = \lim_{h\to 0} \frac{-2xh-h^2}{\left(\sqrt{1-(x+h)^2}+\sqrt{1-x^2}\right)}$$

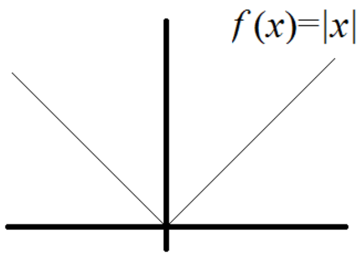
$$\lim_{h\to 0} \left(\frac{h}{h}\right) \frac{-2x-h}{\left(\sqrt{1-(x+h)^2}+\sqrt{1-x^2}\right)} = \frac{-2x}{2\sqrt{1-x^2}} = \frac{-x}{\sqrt{1-x^2}}$$

If you can think back to the first lecture in this class, we derived this using just geometry, and we found the same solution. Hurrah!

Differentiality vs. Continuity

A function can be continuous at some number *x* without being differentiable there.
For example, the absolute value function is continuous everywhere but not differentiable at zero.

If $f(x) = |x|$



$$f'(0) = \lim_{h\to 0} \frac{|0+h|-|0|}{h} = \lim_{h\to 0} \frac{|h|}{h}$$

But from the left this equals -1 and from the right it equals 1, so the limit doesn't exist, and therefore the derivative does not exist at x=0.

What you should notice is that when a function comes to a sharp point, there will be no derivative.

A final useful theorem

If *f* is differentiable at *x*, then *f* is continuous at *x*.

First a Lemma

If *f*(x) is continuous, then $\lim_{x \to a} f(x) = f(a)$

Let *x*=*a*+*h* so $\lim_{a+h \to a} f(a+h) = f(a)$

But clearly as $a + h \to a$ we have $h \to 0$ so,

$\lim_{h \to 0} f(a+h) = f(a)$ This is true for all *x*=*a* so

So *f*(x) is continuous *iff* $\lim_{h \to 0} f(x+h) = f(x)$

Proof

If $h \neq 0$ then $f(x+h)\text{-}f(x) = \frac{f(x+h)\text{-}f(x)}{h} \cdot h$

Finding the limit of both sides we have

$$\lim_{h \to 0}[f(x+h)\text{-}f(x)] = \lim_{h \to 0} \frac{f(x+h)\text{-}f(x)}{h} \cdot \lim_{h \to 0} h$$

With *f* differentiable we have $f'(x) = \lim_{h \to 0} \frac{f(x+h)\text{-}f(x)}{h}$

So, $\lim_{h \to 0}[f(x+h)\text{-}f(x)] = f'(x) \cdot \lim_{h \to 0} h$

But clearly $\lim_{h \to 0} h = 0$.

So, $\lim_{h \to 0}[f(x+h)\text{-}f(x)] = 0$ and therefore

$\lim_{h \to 0}[f(x+h)] = f(x)$

So, by our lemma, *f* (x) is continuous.