

Reciprocal and Quotient Rules

Review

Sum Rule

$$1. (f + g)'(x) = f'(x) + g'(x)$$

The Product Rule

$$2. (f \cdot g)'(x) = f(x) \cdot g'(x) + g(x) \cdot f'(x)$$

The Power Rule

$$3. (x^n)' = n(x^{n-1})$$

Section 3.2

Reciprocal Rule

Assuming that g is differentiable we have

$$4. \left(\frac{1}{g}\right)' = -\frac{g'(x)}{[g(x)]^2} \text{ for } g(x) \neq 0$$

To show this we start with our usual definition of a derivative

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{1}{g(x+h)} - \frac{1}{g(x)} \right] &= \lim_{h \rightarrow 0} \frac{g(x) - g(x+h)}{h} \cdot \frac{1}{g(x)} \cdot \frac{1}{g(x+h)} = \\ &= - \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \cdot \frac{1}{g(x)} \cdot \frac{1}{g(x+h)} = \\ &= -g'(x) \lim_{h \rightarrow 0} \frac{1}{g(x)} \frac{1}{g(x+h)} = -\frac{g'(x)}{[g(x)]^2} \end{aligned}$$

Example:

$$f(x) = \frac{1}{\sqrt{x}} = \frac{1}{g(x)}$$

$$f'(x) = -\frac{g'(x)}{[g(x)]^2} = -\frac{\frac{1}{2\sqrt{x}}}{(\sqrt{x})^2} = -\frac{\frac{1}{2\sqrt{x}}}{x} = -\frac{1}{2\sqrt{x^3}}$$

Example:

Let $n > 0$ and let $p(x) = x^{-n} = \frac{1}{x^n}$ so, we have $g(x) = x^n$

Using the power and reciprocal formulas

$$p'(x) = \left(\frac{1}{g(x)}\right)' = -\frac{g'(x)}{[g(x)]^2} = -\frac{nx^{n-1}}{x^{2n}} = -nx^{-n-1}$$

Note: this shows that the power rule works for n an integer < 0

The Quotient Rule

$$5. \left(\frac{f}{g}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2} \text{ for } g(x) \neq 0$$

I find this the hardest to remember. The key I think is the recall that the first term in the numerator is f' .

To show this rule we use the product and reciprocal rules

$$\left(\frac{f}{g}\right)' = \left[f(x) \frac{1}{g(x)}\right]' = f(x) \left(\frac{1}{g(x)}\right)' + \frac{1}{g(x)} f'(x) =$$

$$f(x) \left(-\frac{g'(x)}{[g(x)]^2}\right) + \frac{f'(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

Example:

$$f(x) = \frac{5x}{1+x}$$

$$f'(x) = \frac{(5x)'(1+x) - (5x)(1+x)'}{(1+x)^2} = \frac{5(1+x) - 5x}{(1+x)^2} = \frac{5}{(1+x)^2}$$

Example

$$f(x) = \frac{6x^2 - 1}{x^4 + 5x + 1}$$

What is $f'(x)$?

Here we must use the quotient rule

$$f'(x) = \frac{(6x^2 - 1)'(x^4 + 5x + 1) - (6x^2 - 1)(x^4 + 5x + 1)'}{(x^4 + 5x + 1)^2} =$$

$$\frac{12x(x^4 + 5x + 1) - (6x^2 - 1)(4x^3 + 5)}{(x^4 + 5x + 1)^2} =$$

$$\frac{12x^5 + 60x^2 + 12x - 24x^5 + 4x^3 - 30x^2 + 5}{(x^4 + 5x + 1)^2} =$$

$$\frac{-12x^5 + 4x^3 + 30x^2 + 12x + 5}{(x^4 + 5x + 1)^2}$$

$$f'(0) = \frac{5}{1} = 5$$