Math 109 Calc 1 Lecture 7

Derivative of the Exponential Function and e

Section 3.1

Derivative of e^x

Consider the function $f(x) = 10^x$

Using the definition of the derivative we have

$$f'(x) = \lim_{h \to 0} \frac{10^{x+h} - 10^x}{h}$$
$$= \lim_{h \to 0} \frac{10^x 10^h - 10^x}{h} = \lim_{h \to 0} \frac{10^x(10^h - 1)}{h} = 10^x \lim_{h \to 0} \frac{(10^h - 1)}{h}$$

Notice that

$$f'(0) = \lim_{h \to 0} \frac{10^{0+h} - 10^0}{h} = \lim_{h \to 0} \frac{10^h - 1}{h}$$

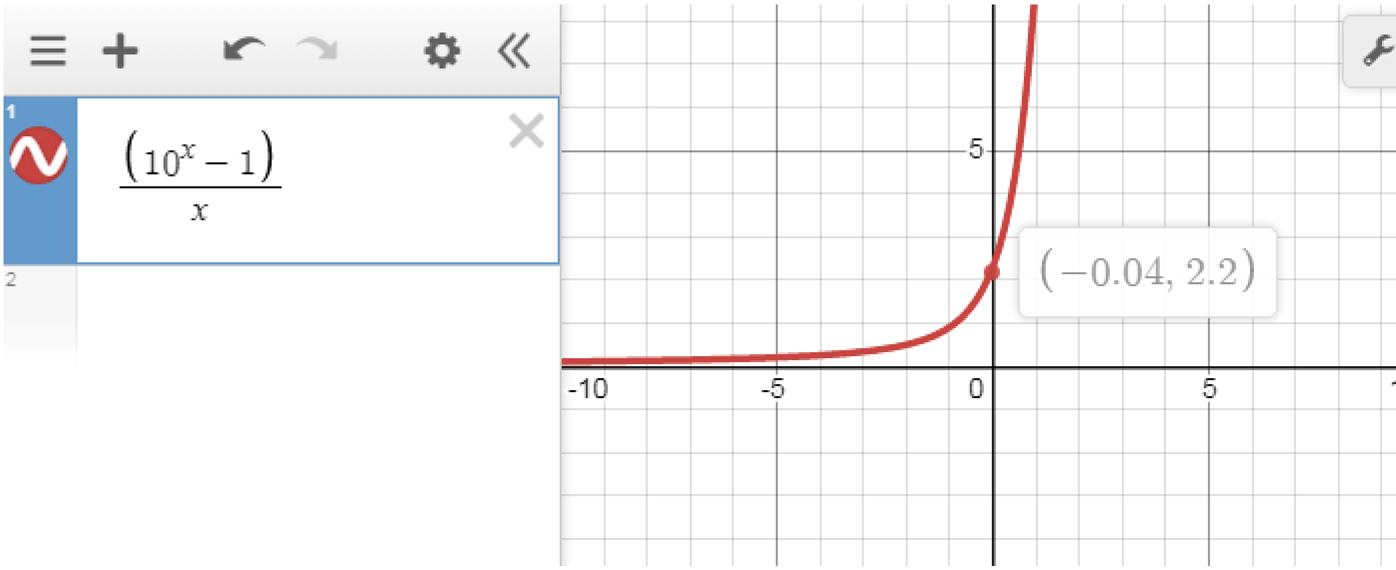
So, we've shown that for $f(x) = a^x$

$$f'(x) = f'(0)f(x)$$

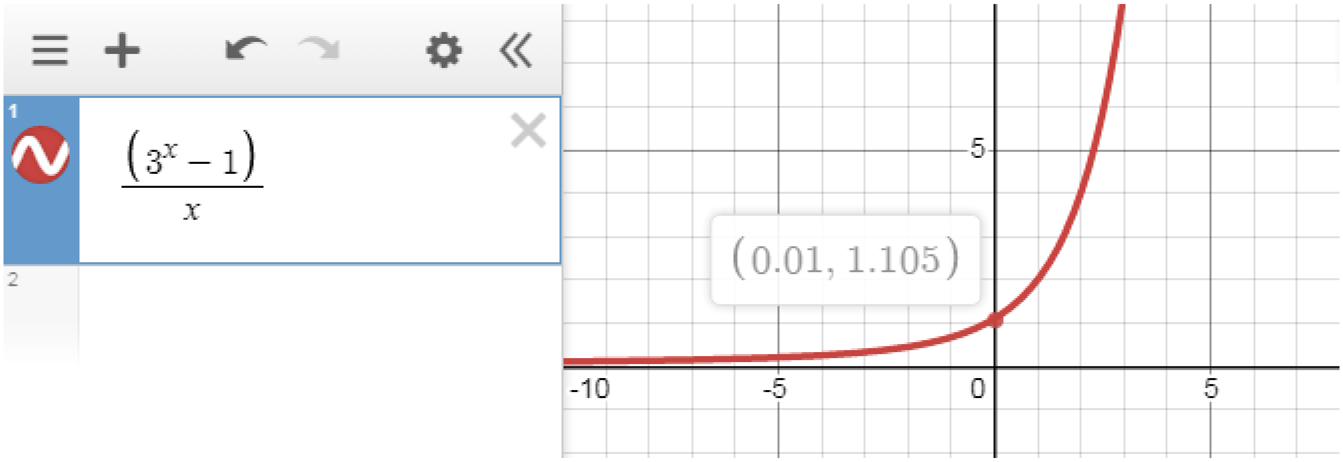
We want to look at what the values of $f'(0)$ are for various values of a. with the goal of find an a such that $f'(0) = 1$.

Using DESMOS.

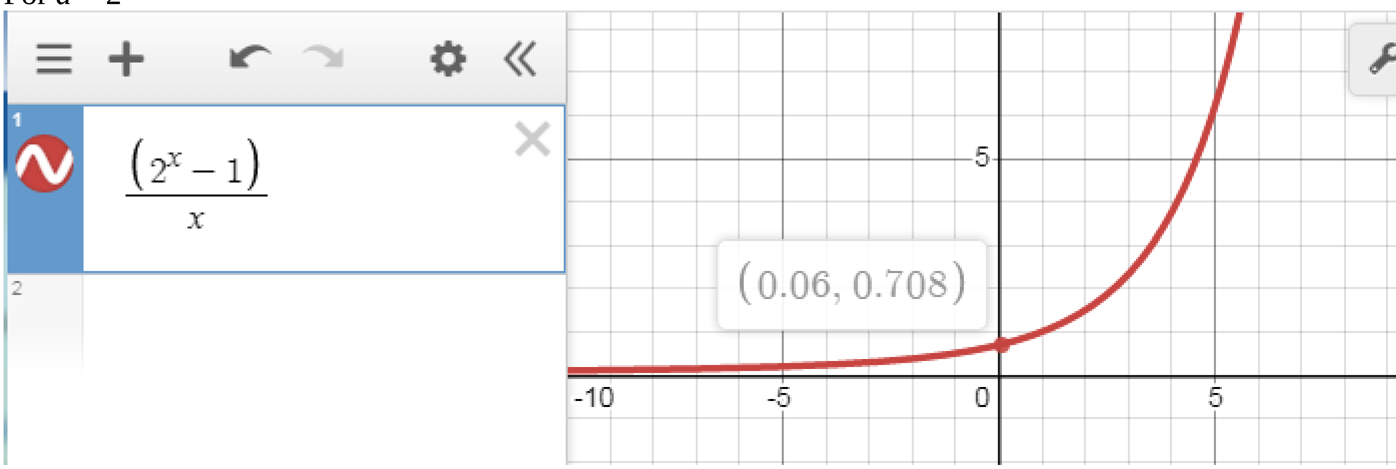
For a = 10


1
$\frac{(10^x - 1)}{x}$
2
5
(−0.04, 2.2)
-10
-5
0
5

For $a = 3$



For $a = 2$



We make a chart

a	$\lim_{h \to 0} \frac{a^h - 1}{h}$
10	2.2
3	1.105
2	.708

We conclude that there must be some number on the interval [2,3] where $\lim_{h \to 0} \frac{10^h - 1}{h} = 1$

We call that number *e* which stands for Euler's constant.

This number is approximately 2.71828

You may have come across this number in another context.

If you are doing finance and you want to see how your principle will grow with time, you have the following equation for the future value FV in terms of the present value PV given an interest rate *i*.

If interest is computed yearly, then the equation is

$$FV = PV(1 + \mathrm{i})^{y}$$

If the interest is computed monthly

$$FV = PV\left(1 + \frac{\mathrm{i}}{12}\right)^{12y}$$

If the interest is computed every day then

$$FV = PV\left(1 + \frac{\mathrm{i}}{365}\right)^{365y}$$

Some banks will compute the interest infinitely in which case we take a limit

$$FV = PV \lim_{n\to\infty}\left(1 + \frac{\mathrm{i}}{n}\right)^{ny} = PV \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^{n(iy)}$$

We can extract the number $\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^{n} = e \approx 2.71828$ the same constant we found above.

The magic to behold here is that we've found a number *e* such that

$$[e^{x}]' = e^{x}$$

This seems like a good time to remind you of the alternate syntax we can use.

$$\frac{d}{dx}e^{x} = e^{x}$$

Occasionally we will want to find the derivative of a derivative and we can use the following syntaxes

$f(x)''$ or $f(x)^{(3)}$

The Leibniz version of this
Could be $\frac{d^2}{dx^2}f(x)$
Or if we consider $y = f(x)$

$$\frac{d^2y}{dx^2}$$