Math 109 Calc 1 Lecture 4

Continuity

Section 2.5: What is Continuity.

First a review of intervals

We use set notation sometimes to describe a set of points.

Set notation has either a list {1, 2, 3} or a description {*x*>0} inside curly brackets.

We might use a fancy ‘r’ $\mathbb{R}$ to mean all real numbers.

We could use this with set notation $\{x \in \mathbb{R}\}$

Or we could put some condition on the values $\{x \in \mathbb{R} : 0 < x < 1\}$

This indicates the **open interval** from 0 to 1.

Note that the $\in$ symbol mean “a member of”

And the ‘:’ means “such that”.

We also can write the interval just mentioned as (0,1).

Note, this looks like but is not a set of *x*, *y* coordinates.

You can only tell the difference from context.

The set $\{x \in \mathbb{R} : 0 \le x \le 1\}$ describes the **closed interval** from 0 to 1.

By closed we mean it includes the end points, while an open set excludes the end points.

A set can be partially open and closed, eg. (0, 1], or [0,1).

The set $\{x \in \mathbb{R} : 0 < x\}$ can be written as $(0, \infty)$.

Note that infinite endpoints are always written as open.

Before we get to the exciting part of this course, differentiation, we need to understand an important mathematical concept, **continuity**.

Continuity in everyday language means something goes on without interruption or abrupt changes. In mathematics the concept is very similar.

In a very rough way, a continuous function is one that one can draw on a graph without removing a pencil from the paper or a marker from the blackboard. Of course, this is not very accurate nor a very mathematical way of talking. You will find that continuity is very bound to our previous subject, limits.

There are three types of continuity that you will need to know about, all related. (Illustrate on the board)

1. Continuity at a point

2. Continuity on an interval

3. Continuity of a function on its domain

We will start with continuity at a point, or pointwise continuity.

A function f is continuous at a point c if and only if the following is true.

$$\lim_{x\to c} f(x) = f(c)$$

Note that there is no requirement for the limit of a function at a point to be equal to the function evaluated at that point. Consider for example:

$$\lim_{x\to 1} \frac{x-1}{x-1}$$

While the function $\lim_{x\to 1} \frac{x-1}{x-1}$ has a limit at 1, the function is not defined there. And since $f(1)$ is not defined, the function is not continuous there.

Note that three things are required for a function to be continuous at a point.

1. The function is defined at the point f (c), so c is in the domain of the function.

2. The limit $\lim_{x \to c} f(x)$ exists

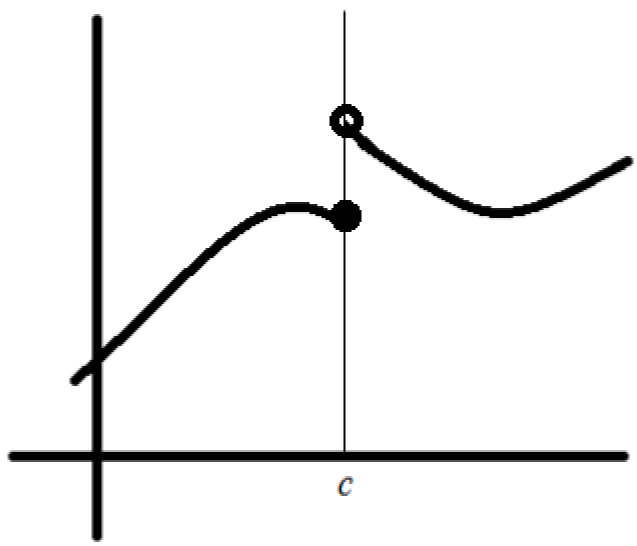
3. The limit and the function evaluated at the limit must be equal.

$$\lim_{x \to c} f(x) = f(c)$$

Failing any of these three the function is not continuous at that point. We call this a discontinuity.

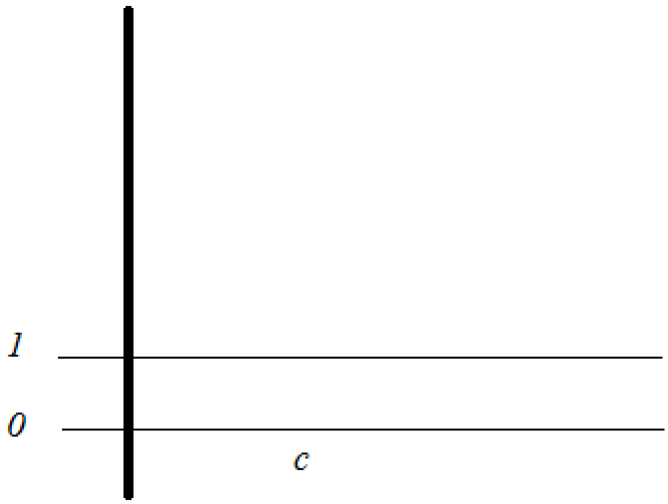
Discontinuous functions:

Here are some examples of a discontinuous function.

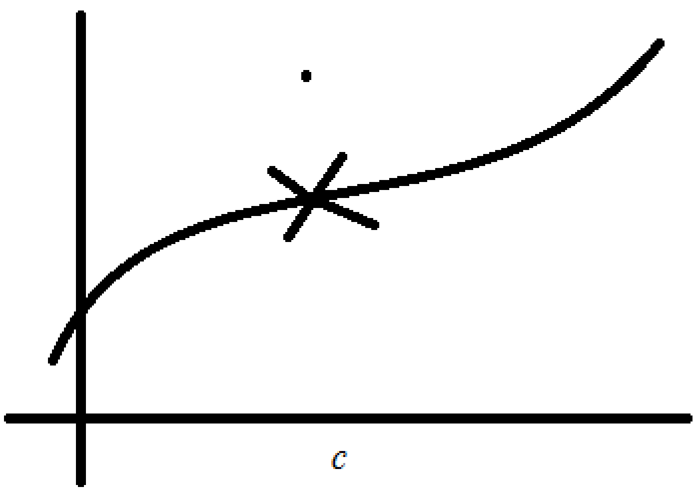


Here the function clearly jumps from one value to another at the point c.
This has a name, a **jump discontinuity**.

$$f(x)=\begin{cases}1, & x \text{ rational} \\ 0, & x \text{ irrational}\end{cases}$$

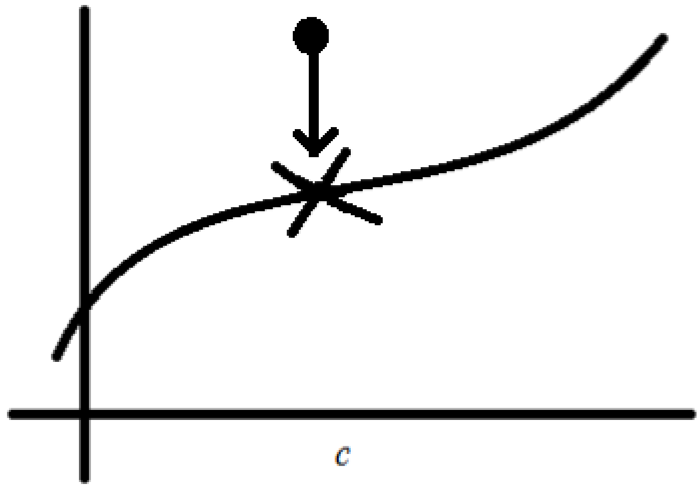


This function, known as the Dirichlet function is not just discontinuous at *c* but it is discontinuous at every point, so we say the function is **discontinuous everywhere**



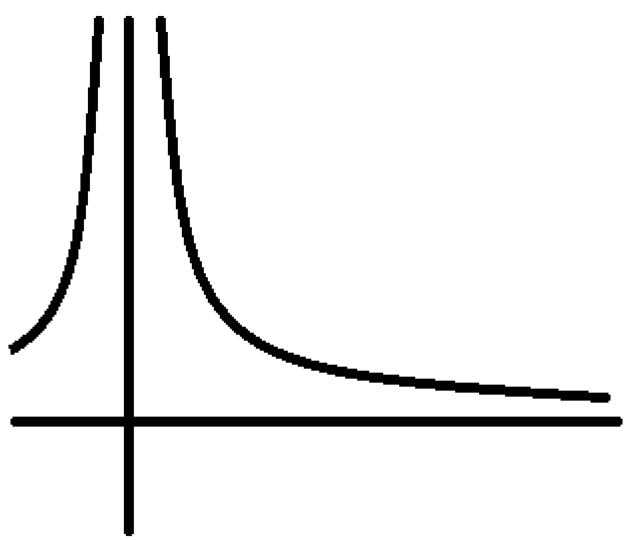
This function just happens to have the point at *c* defined to be not where the limit at *c* is. This type of discontinuity also has a name, it is a **removable discontinuity**.

By just moving the value of the one aberrant point *c* to *f* (c) we can fix the discontinuity and get a function continuous at *c*.



Note that we can fix $\lim_{x \to c} f(x) = \frac{x-1}{x-1}$ by defining $f(1) = 1$

Finally, think about the function $f(x) = \frac{1}{|x|}$



This function has a discontinuity at *c*=0 but it is unlike a jump discontinuity or a removable discontinuity. We call this a discontinuity at infinity.

Examples

Where are these functions discontinuous?

$$f(x) = \frac{x^2 - x - 2}{x - 2} = \frac{(x+1)(x-2)}{x-2}$$

$$f(x) = \begin{cases} \frac{x^2 - x - 2}{x - 2} \text{ if } x \neq 2 \\ 1 \text{ if } x = 2 \end{cases}$$

$$f(x) = \begin{cases} \frac{1}{x^2} \text{ if } x \neq 0 \\ 1 \text{ if } x = 0 \end{cases}$$

Continuity on an interval

The second type of continuity is continuity on an interval.

A function is continuous on an interval if for every c in that interval, the function is pointwise continuous at c.

If you are being picky, you will note that the endpoints of an interval do not have a limit, though they have a one-sided limit, and therefore they have one sided continuity. We will not be picky in this class and will not worry about this technicality.

Saying that **a function is continuous** means that it is continuous at every point in its domain.

Examples of continuous functions
Most of the functions you have encountered so far are continuous. Here are some examples. (Illustrate with Examples)

1. Polynomial functions
2. Absolute value functions
3. Root functions such as $f(x) = \sqrt{x}$
4. Exponential functions
5. Logarithmic functions
6. Rational functions such as $P(x)Q(x)$ on their domain, so, where $Q(x) \neq 0$
7. The trigonometric functions except for where they go to infinity, e.g. $\tan\left(\frac{\pi}{2}\right)$
8. The inverse trigonometric function

Example:

Show that the function $f(x) = 1 - \sqrt{1 - x^2}$ is continuous on the interval (-1,1)
Note that *x* is defined on every value of (-1,1)

If $-1 < a < 1$ then $\lim_{x \to a} f(x) = \lim_{x \to a} 1 - \sqrt{1 - x^2} =$

$$1 - \sqrt{\lim_{x \to a}\left(1 - x^2\right)} = 1 - \sqrt{1 - \lim_{x \to a} x^2} = 1 - \sqrt{1 - a^2} = f(a)$$

So, by the definition of continuity, f(x) is continuous on (-1,1)

Properties of continuous functions

The limit theorems we learned provide some useful theorems for combining continuous functions.

Example:
Since $\lim_{x \to c} K\, f(x) = K \lim_{x \to c} f(x)$
It should be immediately obvious that $\lim_{x \to c} K\, f(x) = K f(c)$
which is to say that if

$f(x)$ is continuous at *c*, then $K\, f(x)$ is continuous at *c*.

Other similar properties concerning continuous functions *f* and *g* are

1. *f*+*g* is continuous
2. *f*-*g* is continuous
3. *fg* is continuous
4. $\dfrac{f}{g}$ is continuous where $g(x) \neq 0$
5. $f(g(x))$ is continuous, composition

Example:

Find $\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x}$

Since $x = -2$ is in the domain of the function,

$$\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x} = f(-2) = \frac{(-2)^3 + 2(-2)^2 - 1}{5 - 3(-2)} = \frac{-8 + 8 - 1}{5 + 6} = \frac{-1}{11} =$$

Find $\lim_{x \to \pi} \frac{\sin x}{2 + \cos x} =$

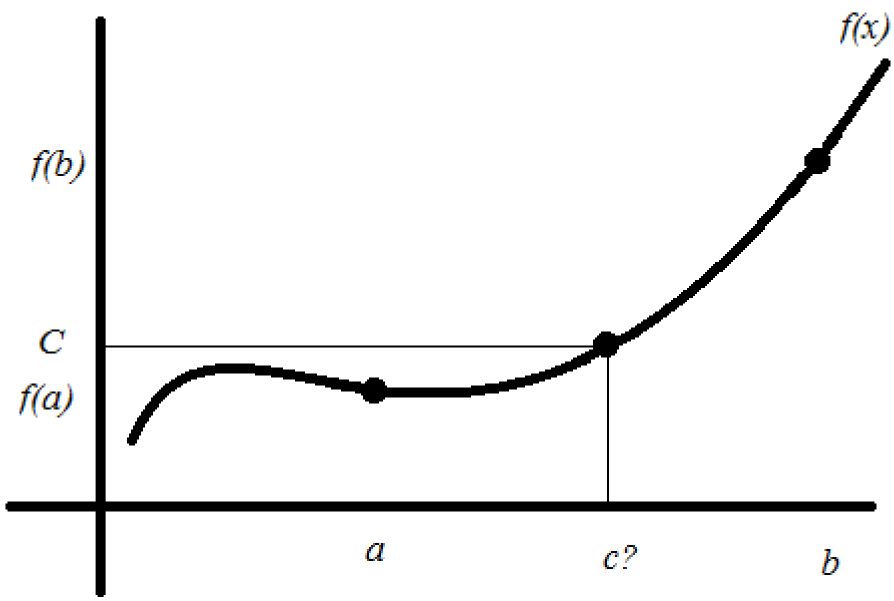
Since *sin x* and *cos x* are continuous, and $2 + \cos(\pi) \neq 0$

$$\lim_{x \to \pi} \frac{\sin x}{2 + \cos x} = \frac{\sin \pi}{2 + \cos \pi} = \frac{0}{1} = 0$$

Two useful theorems related to continuity

The intermediate value theorem

If f is continuous on $[a,b]$ and C is a number between $f(a)$ and $f(b)$, then there is at least one number c between a and b such that $f(c)=C$.



Note that the theorem tells you that c must exist, but it doesn't say anything about how to find it.

Example:

Show that there is a solution to the equation

$$4x^3 - 6x^2 + 3x - 2 = 0$$

on the interval [1,2]

$f(1) = 4 - 6 + 3 - 2 = -2$ which is less than 0

and

since $f(2) = 4(8) - 6(4) + 3(2) - 1 = 13$ which is greater than 0

By the intermediate value theorem there must be a number c between 1 and 2 such that $f(c) = 0$.

The Maximum-Minimum Theorem

If *f* is continuous on [*a*,*b*] then somewhere on [*a*,*b*] *f* takes on a maximum value M and a minimum value m.

Note that these extreme values may be at the end points.

Also note that if is *f* is just continuous on the open interval (a,b) the theorem does not hold.

Example:

Consider for example $tan(x)$ on the interval $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$