Math 109 Calc 1 Lecture 4.5

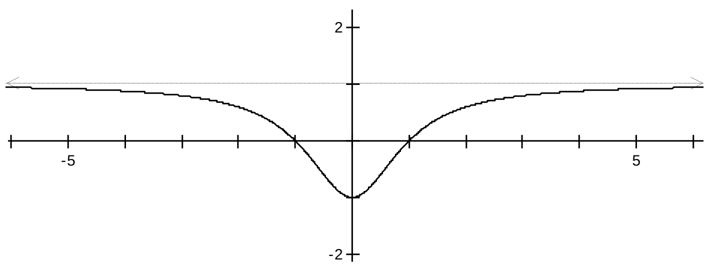
Limits at Infinity

Section 2.6: Limits at Infinity

First, we look at a rational function

$$f(x)=\frac{x^2-1}{x^2+1}$$

The graph of the function looks as follows:



Notice that as *x* gets larger and larger, the function gets closer and closer to the asymptote at y=1. The same happens as *x* gets smaller and smaller.

We have a name for this behavior.

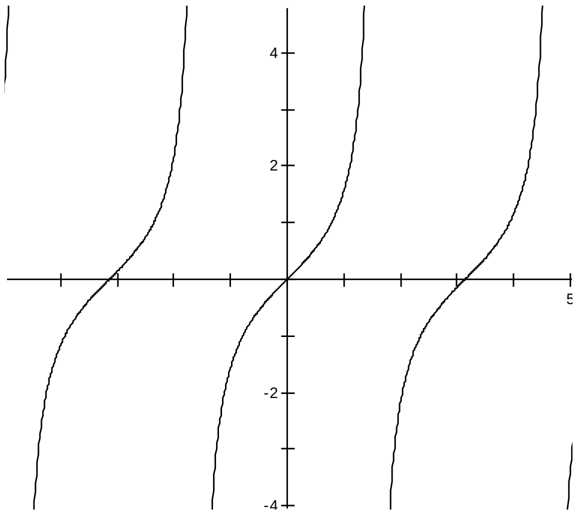
We say that the function has a limit at infinity.

The notation is

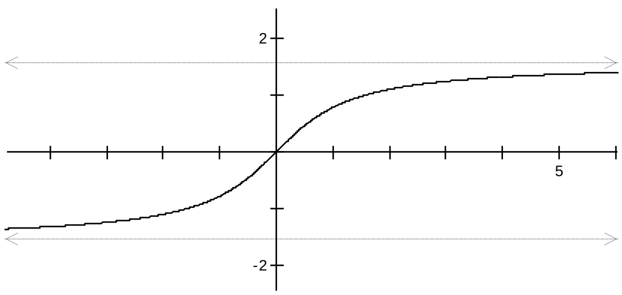
$$\lim_{x\to\infty}\frac{x^2-1}{x^2+1}=1 \text{ and } \lim_{x\to-\infty}\frac{x^2-1}{x^2+1}=1$$

The intuitive idea is that we can get as close to the limit as desired by making *x* large enough.

Recall that the graph of the function $f(x) = \tan(x)$
Looks like this:



We restrict the domain of this function to get an inverse the arctan(x)



Note that

$$\lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2} \text{ and } \lim_{x\to-\infty} \tan^{-1} x = -\frac{\pi}{2}$$

As with a limit to a value of x there is a formal definition for limits at infinity.
We will not cover this definition; however, you may find it in Stewart on page 134.

Instead we will stop with this intuitive understanding and proceed to show some examples on how you evaluate limits at infinity.

STOPPING POINT ON 02/03/2022

A few limits

$$\lim_{x\to\infty} \frac{1}{x} = 0$$

It's easy to see that if that we want to get within some small number ε of 0 we just need to make $x > \frac{1}{\varepsilon}$

$$\lim_{x\to\infty} \frac{1}{x^r} = 0 \quad r > 0$$

The same argument holds for this limit only we just need to make $x > \frac{1}{\sqrt[r]{\varepsilon}}$

Consider $\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1}$

We can evaluate this more easily by dividing the numerator and the denominator by x^2

$$\lim_{x\to\infty} \frac{\frac{3x^2 - x - 2}{x^2}}{\frac{5x^2 + 4x + 1}{x^2}} = \lim_{x\to\infty} \frac{3 - \frac{1}{x} - \frac{2}{x^2}}{5 + \frac{4}{x} + \frac{1}{x^2}}$$

At this point one can use the limit laws to break this up, however it should be obvious at this point that the terms with x in their denominators will all go to zero leaving

$$\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} = \frac{3}{5}$$

The same strategy will work with this limit

$$\lim_{x\to\infty} \frac{\sqrt{2x^2 + 1}}{3x - 5} = \lim_{x\to\infty} \frac{\sqrt{2 + \frac{1}{x^2}}}{3 - \frac{5}{x}} = \frac{\sqrt{2}}{3}$$

In this example

$$\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)$$

We need a new strategy.

We fall back on our knowledge of the pattern $(A+B)(A-B)=A^2-B^2$ to deal with the square root. We do this by multiplying our function by

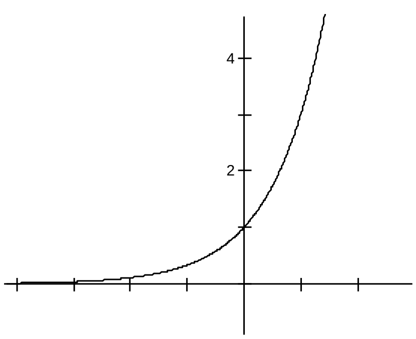
$$\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}$$

This give us

$$\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)\left(\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}\right)=\lim_{x\to\infty}\frac{x^2+1-x^2}{\sqrt{x^2+1}+x}=\lim_{x\to\infty}\frac{1}{\sqrt{x^2+1}+x}$$

Since $\sqrt{x^2+1}+x>x$ we can be sure that $\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)=0$

One last important example is the limit of the exponential function $f(x)=a^x$ where $a>1$ as x goes to negative infinity.



Inspection of any such graph should make it clear that

$$\lim_{x\to-\infty} f(x)=0$$

Limits at Infinity

It should be obvious that not all limits at infinity will exist, for example: $\lim_{x\to\infty} x^2$

The function $f(x) = x^2$ will get larger and larger as *x* gets larger.

While this limit does not exist (DNE) we can indicate the way in which it gets larger as follows:

$$\lim_{x\to\infty} x^2 = \infty$$

Keep in mind that infinity is not a number, but this notation is useful to indicate the direction that the function is going.

This is another example of a limit at infinity.

Other examples that should be obvious are:

$$\lim_{x\to-\infty} x^3 = -\infty$$

$\lim_{x\to\infty} a^x = \infty$ for *a*>1

$$\lim_{x\to\infty} \frac{x^2+x}{3-x} = \lim_{x\to\infty} \frac{\frac{x^2+x}{x}}{\frac{3-x}{x}} = \lim_{x\to\infty} \frac{x+1}{\frac{3}{x}-1}$$

Since the denominator goes to -1 and the numerator goes to infinity $\lim_{x\to\infty} \frac{x^2+x}{3-x} = -\infty$

For this limit we can use a strategy already shown