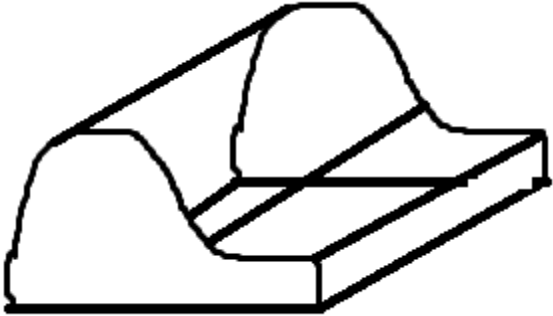


Volume 6.2

There are different strategies for finding volumes using integrals.
If a volume has a fixed cross section:



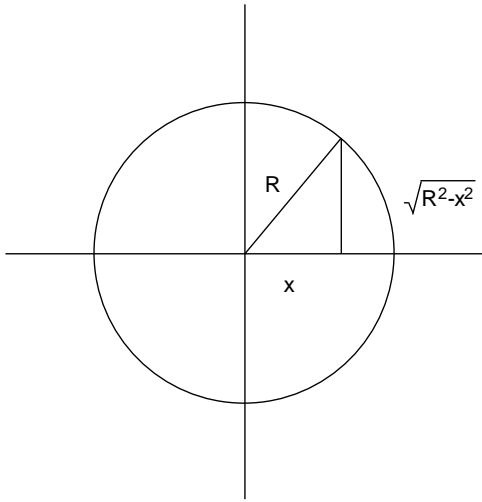
It is merely a matter of finding the area of the of the cross section and multiplying by the length.

$$V = A \cdot L$$

If a volume has a cross section of known area $A(x)$ at each location x then we can find the volume as follows:

$$V = \int_a^b A(x) dx$$

Example 1: Find the volume of a sphere.

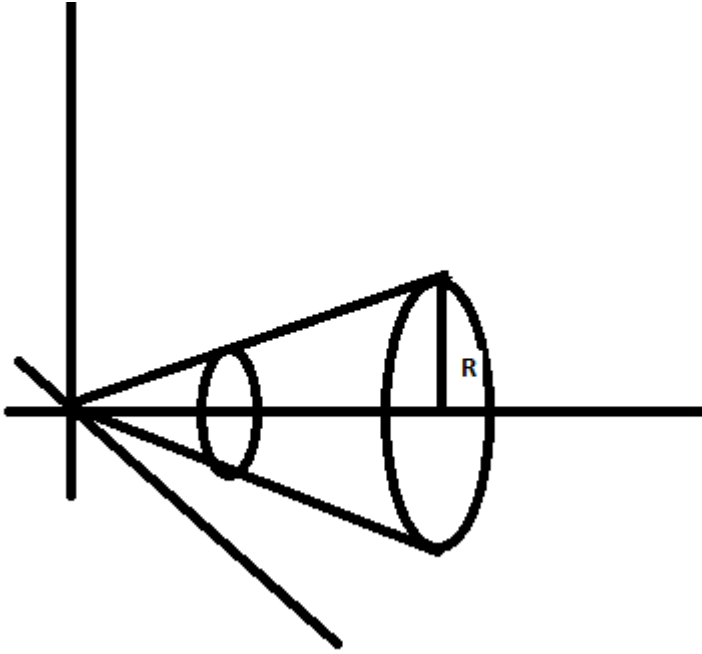


So the radius of the cross section circles is $\sqrt{R^2 - x^2}$
Since the area of a circle is πr^2 the area of each of the
cross sectional circles is $\pi(R^2 - x^2)$

So our volume is

$$V = \int_{-R}^R \pi(R^2 - x^2) dx = \pi \left[R^2 x - \frac{x^3}{3} \right]_{-R}^R = \pi \left[R^3 - \frac{R^3}{3} - \left(-R^3 + \frac{R^3}{3} \right) \right] =$$
$$\pi R^3 \left[2 - \frac{2}{3} \right] = \frac{4}{3} \pi R^3$$

Example 2: Cone base radius R and height L



Here the radius of each circle is $r = \frac{xR}{L}$ so the area of each circle is $A(x) = \pi \frac{x^2 R^2}{L^2}$

The integral/volume

$$V = \int_0^L A(x) dx = \int_0^L \pi \frac{x^2 R^2}{L^2} dx = \pi \frac{R^2}{L^2} \left[\frac{x^3}{3} \right]_0^L = \pi \frac{R^2}{L^2} \left[\frac{L^3}{3} \right] = \frac{1}{3} \pi R^2 L$$

Example 3:

Take the function $f(x) = \sqrt{x}$ and spin it around the X axis forming a solid. What is the volume of this solid with respect to it's height.

We have $V = \int_0^H A(x) dx$ where

$$A(x) = \pi r^2 = \pi (\sqrt{x})^2 = \pi x$$

$$\text{So } V = \pi \int_0^H x dx = \pi \left[\frac{x^2}{2} \right]_0^H = \frac{\pi H^2}{2}$$

Example 4: Rotating around the Y axis.

Let $f(x) = x^3$ be rotated around the Y axis on the interval $[0,8]$

$$V = \int_0^8 A(y) dy$$

But with $y = x^3$ we have the radius $x = y^{1/3}$.

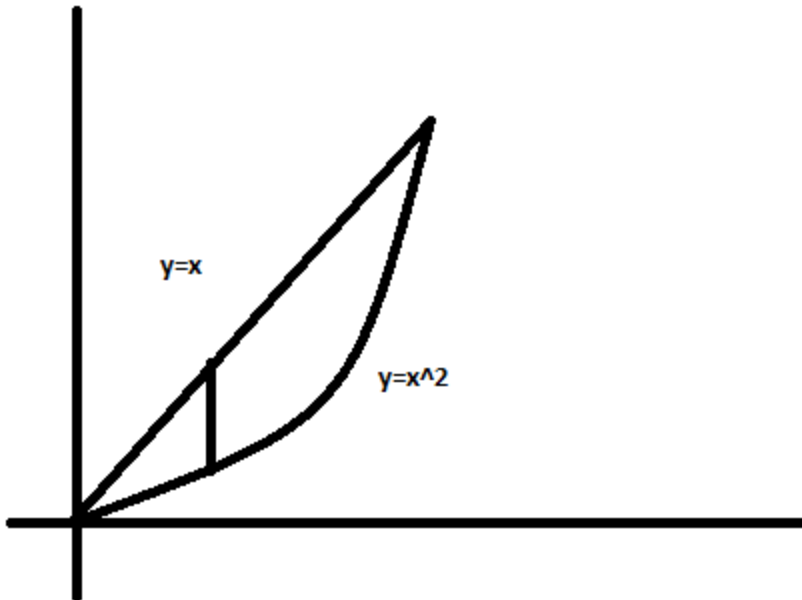
The area of the circles are then $A(y) = \pi x^2 = \pi y^{2/3}$ so

$$V = \pi \int_0^8 y^{2/3} dy = \pi \left[\frac{y^{5/3}}{5/3} \right]_0^8 = \frac{3\pi}{5} [32 - 0] = \frac{96\pi}{5}$$

Example 4: "Washer"

Let the area between $y = x$ and $y = x^2$ be spun around the x axis.

The area is now the area of an annulus or a ring, sometimes known as a washer.



The area function is $A(x) = \pi x^2 - \pi (x^2)^2$

Setting the two functions equal we find the points of intersection:

$$x = x^2$$

$$x^2 - x = 0$$

$$x(x-1) = 0$$

So we want to integrate as follows:

$$V = \pi \int_0^1 (x^2 - x^4) dx = \pi \left[\frac{x^3}{3} - \frac{x^5}{5} \right]_0^1 = \pi \left(\frac{1}{3} - \frac{1}{5} \right) = \frac{2\pi}{15}$$

Hand out Worksheet,

If time permits go over review sheet.