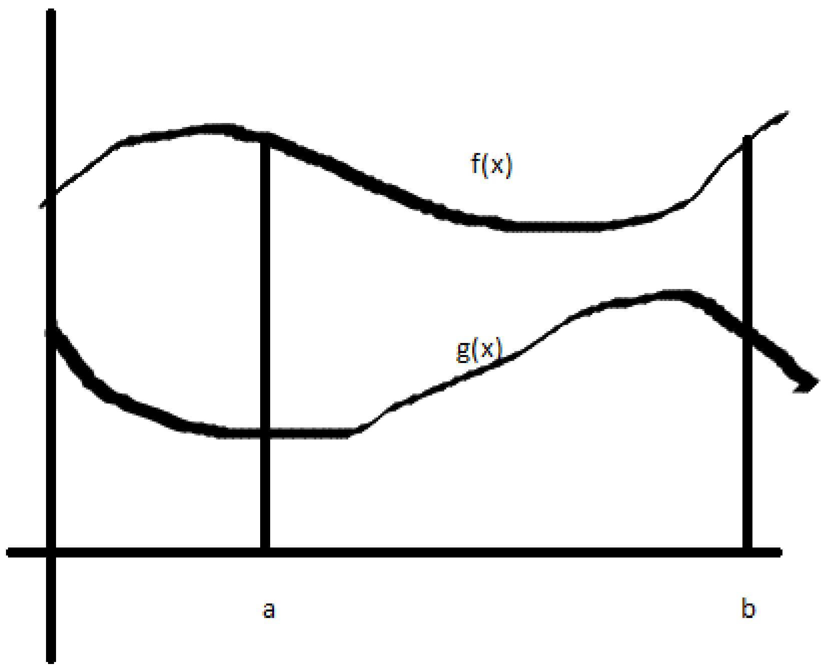
Math 109 Calc 1 Lecture 34

Area Between Curves 5.6

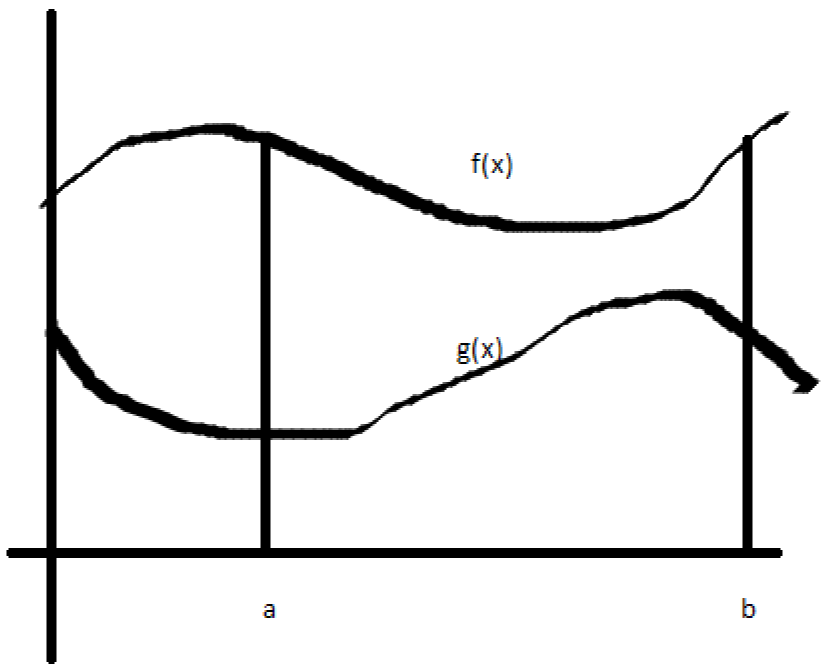
We've been looking at a definite integral as the area beneath a curve, that is the area between the curve and $y=0$.

If the y coordinate of the curve is < 0 we treat this as negative area.
What about the area between two curves?



Clearly the area below $f(x)$ minus the area below $g(x)$ is the area between the curves.

What if one or both functions drop below the X axis?

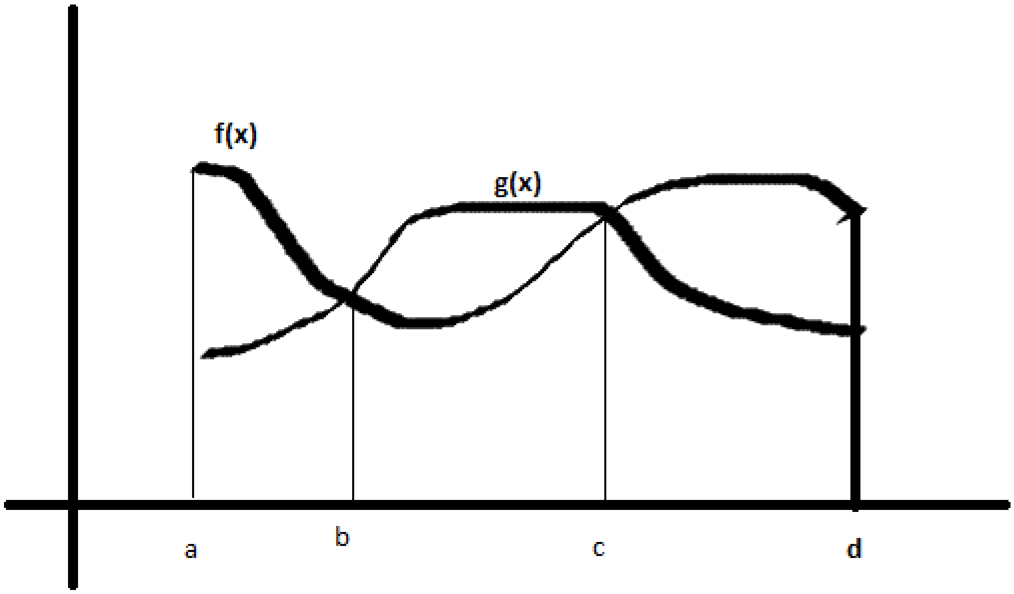


We can add a constant amount to both functions, moving them up above the line preserving the area. Then:

$$\int_a^b f(x)dx + C - \left[\int_a^b g(x)dx + C \right] = \int_a^b f(x)dx - \int_a^b g(x)dx + \int_a^b C\, dx - \int_a^b C\, dx =$$

$$\int_a^b f(x)dx - \int_a^b g(x)dx$$

What if we have two functions that cross over and we want all the area between them?



Then we need to calculate

$$\int_a^d |f(x)-g(x)|dx = \int_a^b f(x)-g(x)dx - \int_b^c f(x)-g(x)dx + \int_c^d f(x)-g(x)dx \int_a^b$$

Example 2: Find the area enclosed by $y = x^2$ and $y = 2x - x^2$

Setting these equal we find

$$x^2 = 2x - x^2$$

$$2x^2 - 2x = 0$$

$$x(x-1) = 0$$

So the points of intersection are 0 and 1.

We integrate $\int_0^1 \left|2x - x^2 - \left(x^2\right)\right| = \int_0^1 \left|2x - 2x^2\right| = \left|\left[x^2 - \frac{2x^3}{3}\right]_0^1\right| = \left|1 - \frac{2}{3} - (0-0)\right| = \frac{1}{3}$

Example 3: Find the area bounded by $y = e^x$ and $y = x$ on the interval [0,1]

Since $e^x > x$ for all $x \in [0,1]$

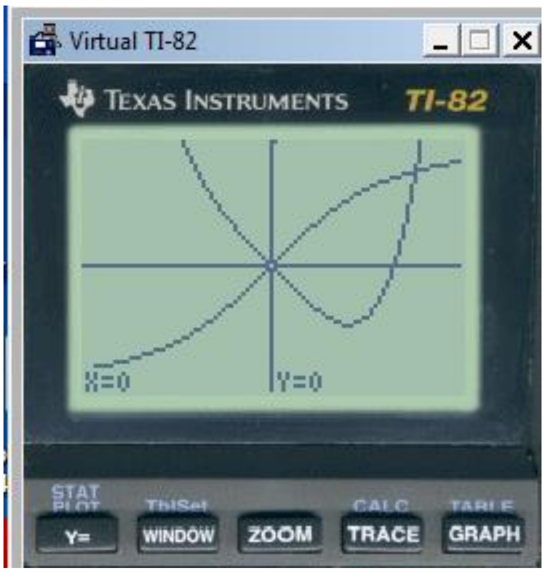
$$A = \int_0^1 \left(e^x - x\right) dx = \left[e^x - \frac{x^2}{2}\right]_0^1 = e - \frac{1}{2} - (1-0) = e - \frac{3}{2}$$

Example 3:

Sometimes it's difficult or impossible to find the points of intersection between curves, but we can approximate them with the calculator:

$$y = \frac{x}{\sqrt{x^2+1}} \qquad y = x^4 - x$$

Clearly (0,0) is a point of intersection.



Using the calculator's Calc/5:Intersection key we find the
other intersection at 1.18

$$\int_0^{1.18} \frac{x}{\sqrt{x^2+1}} - \left(x^4 - x\right) dx$$

We could solve this using anti-derivatives, but since it is already an approximation, use the Calc/7:Integration function giving .78538855

Example 4: from the book, We have the speed of two cars at equal intervals

t	0	2	4	6	8	10	12	14	16
v_A	0	34	54	67	76	84	89	92	95
v_B	0	21	34	44	51	56	60	63	65
v_A - v_B	0	13	20	23	25	28	29	29	30

How can we calculate the integral over v_A - $v_{B?}$

Simpson's Rule?

What does this "Area" that we are calculating represent?

Sometimes it is easier to integrate along Y instead of X?

Example 5: Find the area between $y = x - 1$ and $y^2 = 2x + 6$

Already we have a problem because the 2nd equation is not a function.

But we can switch x and y

$x = y - 1$ and $x^2 = 2y + 6$ or

$y = x + 1$ and $y = \dfrac{x^2 - 6}{2}$

We find the intersection $x + 1 = \dfrac{x^2 - 6}{2}$

$x^2 - 2x - 8 = 0$

$(x-4)(x+2) = 0$

So

$$\int_{-2}^{4} x + 1 - \left(\frac{x^2 - 6}{2}\right) dx = \int_{-2}^{4} -\frac{x^2}{2} + x + 4\, dx = \left[-\frac{x^3}{6} + \frac{x^2}{2} + 4x\right]_{-2}^{4} =$$

$$-\frac{64}{6} + 8 + 16 - \left(\frac{-8}{6} + 2 - 8\right) = 18$$

Integrating using parametric equations.

Let's say we have a function F(x) which we can't be described in terms of x but we have parametric equations.

$x = g(t) \qquad y = f(t)$

so $y = F\left(g(t)\right) = f(t)$

The substitution rule tells us

$$\int_{\alpha}^{\beta} F\left(g(t)\right)g'(t)\,dt = \int_{g(\alpha)}^{g(\beta)} F(x)\,dx$$

Example 6: The area under one cycle of a cycloid, $2\pi r$.

First Use calculator to show that for parametric equations
$x = \theta - \sin\theta$ and $y = 1 - \cos\theta$ over $[0, 2\pi]$ that the integral is 3π .

$x = g(\theta) = r(\theta - \sin\theta) \qquad y = f(\theta) = r(1 - \cos\theta)$

At $\theta = 0$ and $\theta = 2\pi$ we have $(0,0)$ and $(2\pi r, 0)$

$$\int_0^{2\pi r} F(x)\,dx = \int_0^{2\pi} f(\theta)g'(\theta)\,d\theta = \int_0^{2\pi} r(1-\cos\theta)r(1-\cos\theta)\,d\theta =$$

so $r^2\int_0^{2\pi} 1 - 2\cos\theta + \cos^2\theta\,d\theta = r^2\int_0^{2\pi} 1 - 2\cos\theta + \frac{1+\cos 2\theta}{2}\,d\theta =$

$$r^2\left[\frac{3}{2}\theta - 2\sin\theta + \frac{\sin 2\theta}{4}\right]_0^{2\pi} = r^2\left[\frac{3}{2}2\pi - 0 + 0 - (0 - 0 + 0)\right] = 3\pi r^2$$