Math 109 Calc 1 Lecture 32

Indefinite Integrals 5.4

If we have a function $f(x)$ and its antiderivative $F(x)$

recall that $F'(x) = f(x)$ and $\int_a^b f(x)\, dx = F(b) - F(a) = \left[F(x)\right]_a^b$

It is handy to have some notation to refer to this function $F(x)$ which we call an indefinite integral.

$$F(x) = \int f(x)\,dx$$

Unlike the definite integral, this is not a number. This is a function of *x* or rather a family of functions.

For example:

$\int x\,dx = \frac{x^2}{2} + C$ where *C* is any constant $C \in \mathbb{R}$

The connection between an indefinite integral and a definite integral is that a definite integral can be found by evaluating an indefinite integral at its end points.

$$\int_a^b f(x)dx = \left[\int f(x)dx\right]_a^b$$

We already know a wide variety of antiderivatives, which I will summarize

$\int x^n dx = \frac{x^{n+1}}{n+1} + C$ for $x \neq -1$	$\int \frac{1}{x} dx = \ln\lvert x\rvert + C$
$\int e^x dx = e^x + C$	
$\int \sin(x) dx = -\cos(x) + C$	$\int \cos(x) dx = \sin(x) + C$
$\int \sec^2(x) dx = \tan(x) + C$	$\int \csc^2(x) dx = -ctn(x) + C$
$\int \sec^2(x)\tan(x) dx = \sec(x) + C$	$\int \csc^2(x) ctn(x) dx = -\csc(x) + C$
$\int \frac{1}{x^2+1} dx = \tan^{-1}(x) + C$	$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1}(x) + C$
$\int \sinh(x) dx = \cosh(x) + C$	$\int \cosh(x) dx = \sinh(x) + C$

Some formula's and patterns that will be useful for evaluating definite integrals.

$$\int cf(x) dx = c\int f(x) dx$$

$$\int k\ dx = kx + C$$

$$\int f(x) + g(x) dx = \int f(x) dx + \int g(x) dx$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \text{ for } x \neq -1$$

As a check $\frac{d}{dx}\frac{x^{n+1}}{n+1} + C = x^n + 0 = x^n$

An Important pattern

$\int \frac{f^{'}(x)}{f(x)}dx = \ln\left|f(x)\right| + C$ This comes up a lot, e.g.

$$\int \frac{2x+1}{x^2+x+4}dx$$

Since $\frac{d}{dx}\left(x^2+x+4\right) = 2x+1$

$$\int \frac{2x+1}{x^2+x+4}dx = \ln\left|x^2+x+4\right| + C$$

Another pattern

Since $\frac{d}{dx} f(x)^{-n} = (-n) f(x)^{-n+1} f^{'}(x) = -n\frac{f^{'}(x)}{f(x)^{n-1}}$

we have $\int \frac{f^{'}(x)}{f(x)^n}dx = -\left(\frac{1}{n-1}\right)\frac{1}{f(x)^{n-1}} + C$

Example:

$$\int \frac{x}{\left(x^2+1\right)^2}dx$$

We know that $\frac{d}{dx}\left(x^2+1\right) = 2x$

So we have $\int \frac{x}{\left(x^2+1\right)^2}dx = \frac{1}{2}\int \frac{2x}{\left(x^2+1\right)^2}dx = \frac{1}{2}\left(-\frac{1}{1}\right)\frac{1}{x^2+1} + C = \left(-\frac{1}{2}\right)\frac{1}{x^2+1}$

Example, find: $\int \sec(x)\tan(x)\,dx$, not using the obvious antiderivative:

$$\int \sec(x)\tan(x)dx = \int \frac{1}{\cos(x)}\cdot\frac{\sin(x)}{\cos(x)}dx = \int \frac{\sin(x)}{\cos(x)^2}dx = -\int \frac{-\sin(x)}{\cos(x)^2}dx$$

But $\left[\cos(x)\right]^{'} = -\sin(x)$

So we have $\int \sec(x)\tan(x)dx = -\left(-\frac{1}{\cos(x)}\right) + C = \frac{1}{\cos(x)} + C = \sec(x) + C$

Yet Another Pattern

Recall that $\frac{d}{dx}\sin^{-1}(x)=\frac{1}{\sqrt{1-x^2}}$

Quick Review, how do we know this?

$$\frac{dy}{dx}=\frac{1}{\frac{dx}{dy}}$$

If $x=\sin(y)$ then $\frac{dx}{dy}=\cos(y)$

$$\frac{dy}{dx}=\frac{1}{\cos(y)}=\frac{1}{\sqrt{1-\sin^2(y)}}=\frac{1}{\sqrt{1-x^2}}$$

So clearly, we have $\int\frac{1}{\sqrt{1-x^2}}dx=\sin^{-1}(x)+C$

But likewise, $\int\frac{f'(x)}{\sqrt{1-f(x)^2}}dx=\sin^{-1}(f(x))+C$

Example: $\int\frac{x}{\sqrt{1-x^4}}dx$

Since $\frac{d}{dx}x^2=2x$

$$\int\frac{x}{\sqrt{1-x^4}}dx=\frac{1}{2}\int\frac{2x}{\sqrt{1-\left(x^2\right)^2}}dx=\frac{1}{2}\sin^{-1}\left(x^2\right)+C$$

A similarly useful formula

$$\int\frac{f'(x)}{1+f(x)^2}dx=\tan^{-1}(x)+C$$

Net Change

We can rewrite

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

as

$$\int_a^b F'(x) dx = F(b) - F(a)$$

Where $F'(x)$ is the rate of change of $F(x)$

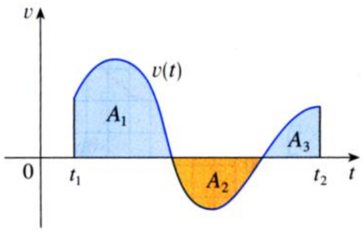
Example:

Consider a reservoir which has water flowing into or out of it at a rate of $V'(t)$

So $V(t_2) - V(t_1) = \int_{t_1}^{t_2} V'(t) dt$ is the change in the amount of water between time t_1 and t_2.

Displacement vs. distance traveled.

Consider a train that travels back and forth on a straight rail at a velocity according to this graph.



If you wish to know its displacement, then

displacement = $A_1 - A_2 + A_3 = \int_{t_1}^{t_2} v(t)\,dt$

If instead you want to know the distance traveled, then you want

distance = $A_1 + A_2 + A_3 = \int_{t_1}^{t_2} |v(t)|\,dt$

Example:

A particle has velocity $v(t)=t^2-t-6$

Find the displacement and distance traveled between 1 and 4 seconds:

displacement = $\int_1^4 \left(t^2-t-6\right)dt =$

$$\left[\frac{t^3}{3}-\frac{t^2}{2}-6t\right]_1^4 = \frac{64}{3}-8-24-\left(\frac{1}{3}-\frac{1}{2}-6\right) = -\frac{9}{2}$$

distance = $\int_1^4 \left|t^2-t-6\right| dt$

Find where the direction changes: $t^2-t-6=(t-3)(t+2)=0$

So it changes at -2 and 3 seconds.

$$\int_1^4 \left|t^2-t-6\right| dt = \int_1^3 \left|t^2-t-6\right| dt + \int_3^4 \left|t^2-t-6\right| dt =$$

$$\left|\left[\frac{t^3}{3}-\frac{t^2}{2}-6t\right]_1^3\right| + \left|\left[\frac{t^3}{3}-\frac{t^2}{2}-6t\right]_3^4\right| =$$

$$\left|9-\frac{9}{2}-18-\left(\frac{1}{3}-\frac{1}{2}-6\right)\right| + \left|\frac{64}{3}-8-24-\left(9-\frac{9}{2}-18\right)\right| =$$

$$\left|-\frac{22}{3}\right| + \left|\frac{17}{6}\right| = \frac{44}{6}+\frac{17}{6} = \frac{61}{6}$$

Try in Class

$$\int_{1}^{4} \sqrt{x}\, dx$$

$$\int_{-1}^{3} (x+1)(x-1)\, dx$$

$$\int_{-2}^{3} \left|(x+1)(x-1)\right|\, dx$$

$$\int \frac{\cos(x)}{\sin(x)^5}\, dx$$

$$\int \frac{t^5 - 4t^3 + t^2 - 8t + 5}{t^2}\, dt$$

$$\int \frac{e^x}{\sqrt{1-e^{2x}}}\, dx$$