Math 109 Calc 1 Lecture 30

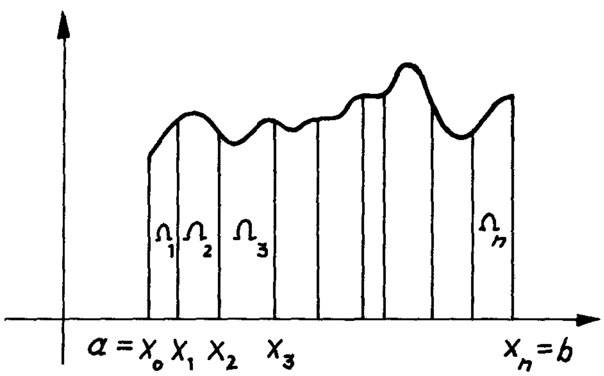
5.2 Definite Integrals

We saw last Thursday that we can describe the area under a curve by the limit of a sum. The book describes this sum slightly differently:

$$\lim_{x\to\infty}\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x$$

Where

$x_i^* \in \left[x_{i-1}, x_i\right]$ and $\Delta x = x_i - x_{i-1}$



We call this the definite integral of *f* from *a* to *b* and write it

$$\int_a^b f\left(x\right) dx$$

If the limit exists we say that *f* is integrable on [*a*,*b*]

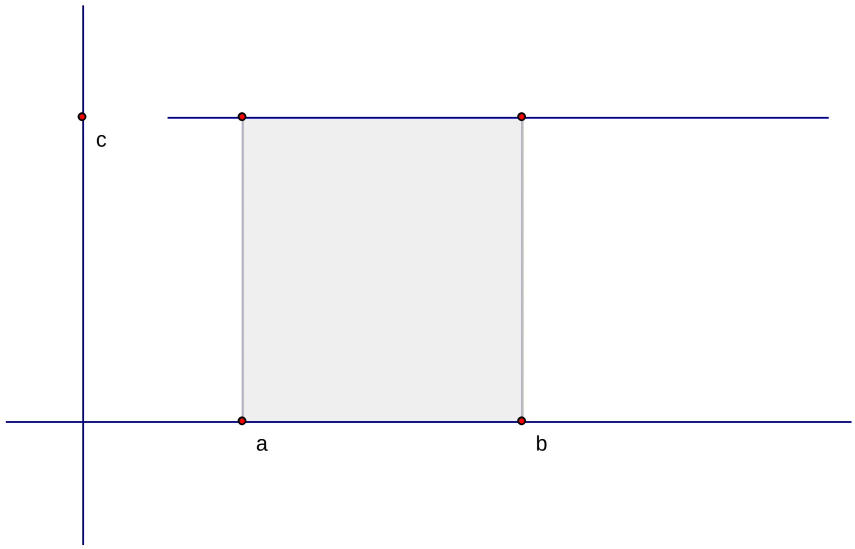
We will find that this definite integral can be used to find the area under a curve as well as volumes of solids and the lengths of curves.

Again, please note that a definite integral is a number.

The variable *x* is just a dummy variable that disappears when you evaluate the integral.

Some simple examples of calculating the area under a curve

$f(x) = c$

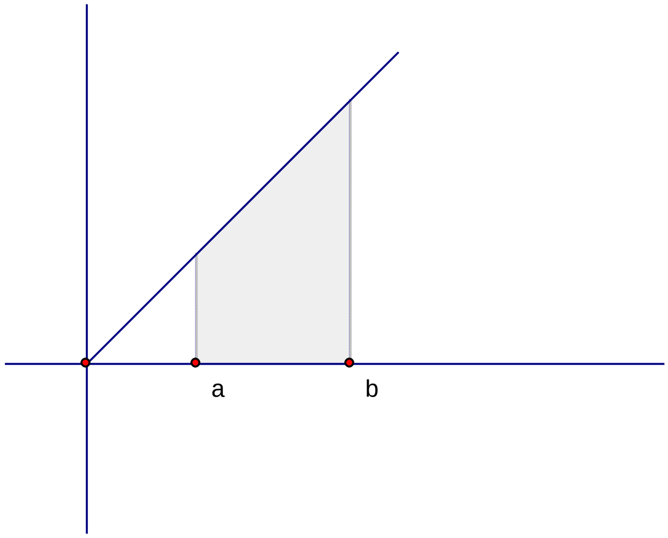


Note that the area under the curve is $A = (b-a)c = bc - ac$

We can write this using the notation $xc\big|_a^b$ or $[xc]_a^b$ which indicates that you evaluate the *xc* at the upper limit and subtract the *xc* evaluated at the lower limit.

Note that we could also have written this as $[F(x)]_a^b$ where $F'(x) = f(x)$

We try this again for a slightly more complex function $f(x) = x$



Here the area can be seen as the difference in area of the two triangles at points:
{(0,0), (a,0), (a,*f*(a))} and
{(0,0), (b,0), (b,*f*(b))}

$$A = \frac{b^2}{2} - \frac{a^2}{2} = \left[\frac{x^2}{2}\right]_a^b$$

Note that we could also have written this as $\left[F(x)\right]_a^b$ where $F'(x) = f(x)$

On Thursday we introduced

$$\int_a^b f(x)dx$$

We now would like to investigate how we might calculate this function in a more direct and exact manner than before.

The preceding examples suggest the possibility that in general

$$\int_a^b f(x)dx = \left[F(x)\right]_a^b \quad = F(b) - F(a)$$

$$Where\ F'(x) = f(x)$$

$$F(x) being\ the\ Anti-Derivative\ of\ \ f(x)$$

We will see that this is the case.

To show this, we proceed by defining a function as follows:

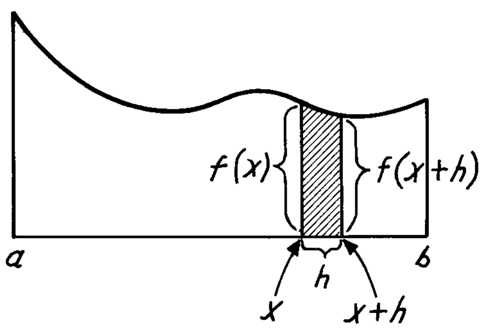
$$F(x) = \int_a^x f(t)\,dt$$

Notice that this is a function of *x* and not a definite integral.
It is a function which simply indicates the area under the curve *f*(*x*) from the point *a* the unknown point *x*.

Now consider this limit, which should look familiar:

$$\lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$$

What does that look like graphically?



$$F(x+h) = \text{area from } a \text{ to } x+h$$
$$F(x) = \text{area from } a \text{ to } x$$
$$F(x+h) - F(x) = \text{area from } x \text{ to } x+h$$
$$\frac{F(x+h) - F(x)}{h} = \frac{\text{area from } x \text{ to } x+h}{h} \overset{\text{approx}}{=} f(x) \text{ for small } h.$$

In this diagram you can see that as *h* --> 0 the shaded area comes closer and closer to being a rectangle with area $\frac{f(x+h)+f(x)}{2}h$

As such

$$\lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h \to 0} \frac{f(x+h)+f(x)}{2} = f(x)$$

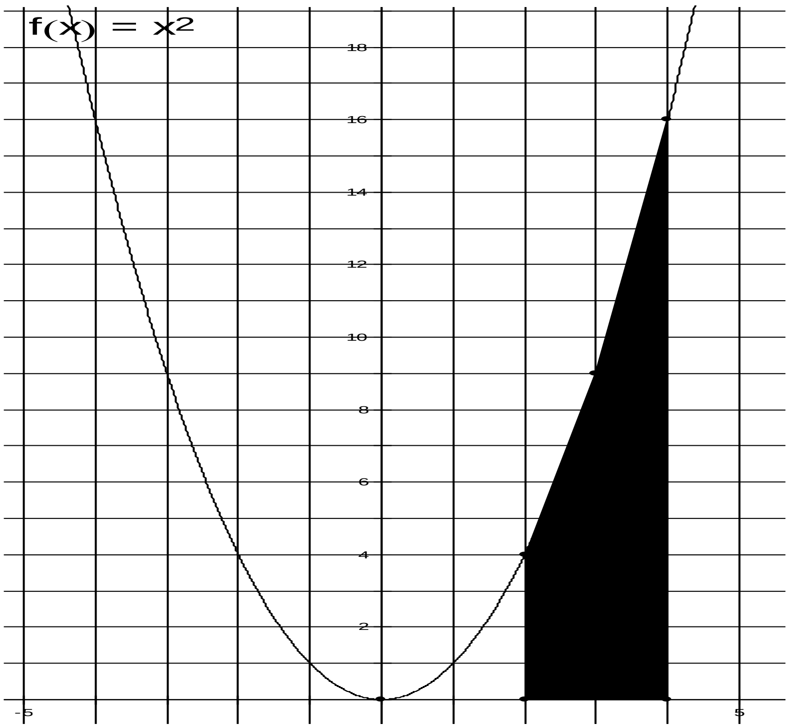
By the definition of the derivative, that means that

$$F'(x) = f(x)$$

That is *F*(*x*) is the anti-derivative of *f*(*x*)

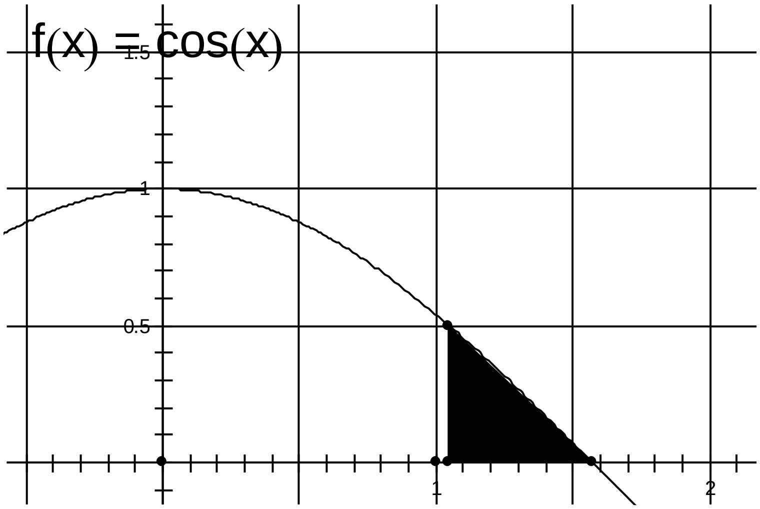
Let's let that settle in a bit with a few examples:

What is the area beneath the function $y = x^2$ between 2 and 4?



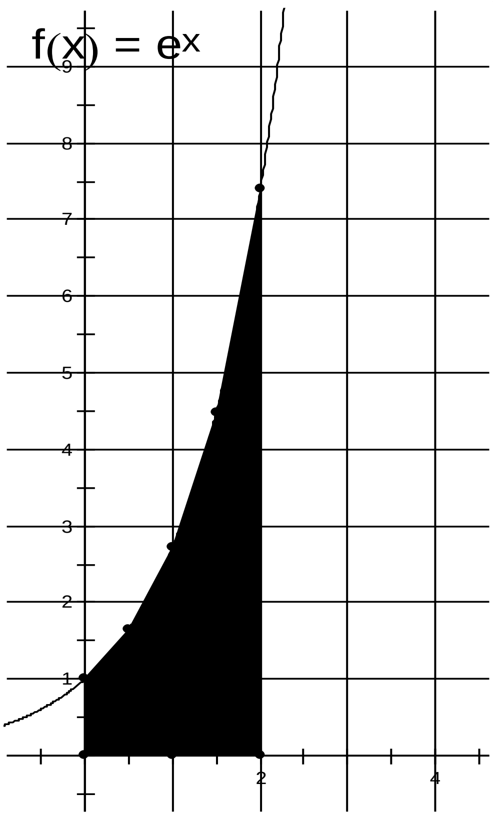
$$\int_2^4 x^2 dx = \left[\frac{x^3}{3}\right]_2^4 = \frac{64}{3} - \frac{8}{3} = \frac{56}{3}$$

What is the area beneath the function $y = \cos(x)$ between $\frac{\pi}{3}$ and $\frac{\pi}{2}$?



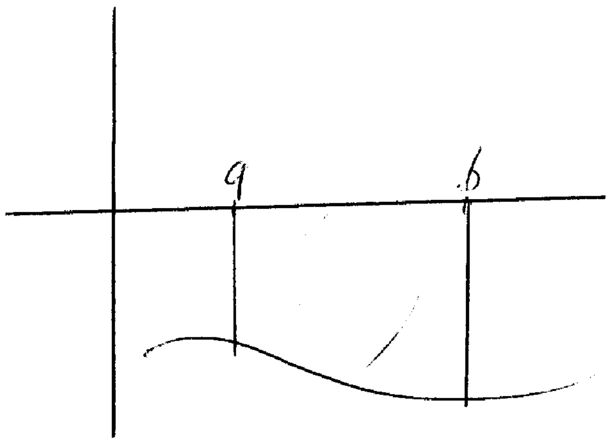
$$\int_{\pi/3}^{\pi/2} \cos(x)\,dx = \left[\sin(x)\right]_{\pi/3}^{\pi/2} = \sin(\pi/2) - \sin(\pi/3) = \frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2} = \frac{\sqrt{2}-\sqrt{3}}{2}$$

What is the area beneath the function $y = e^x$ between 0 and 2?



$$\int_0^2 e^x dx = \left[e^x\right]_0^2 = e^2 - e^0 = e^2 - 1$$

Also note that if $f(x) < 0$, the area is negative:
What happens now if our function is below zero?

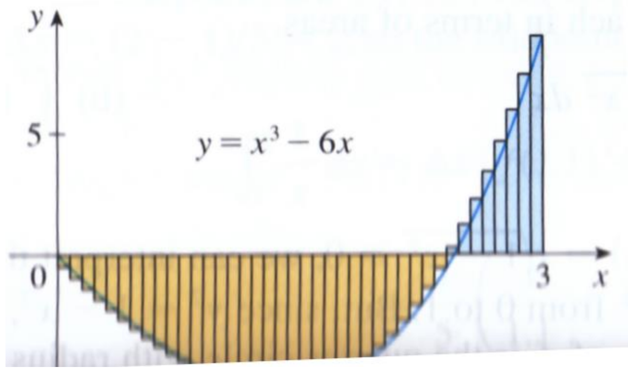


If we go back to our Riemann Sum definition

$$\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x_i$$

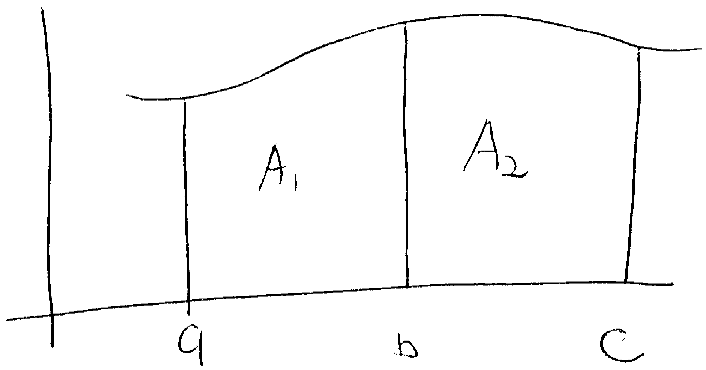
We can see that the result of this sum is now negative. It no longer represents the area, but it is the negative of the area between the function and y=0.

It is also possible that our function is both below and above the x axis.



Here the definite integral might be positive or negative depending on the limits.

Properties of an Integral



If we have $A_1 = \int_a^b f(x)dx$ and $A_2 = \int_b^C f(x)dx$

then is follows that since $A = A_1 + A_2$

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$$

Now re-arranging the limits, we can have

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

But that means that:

$$A_1 = A_1 + A_2 + \int_c^b f(x)dx$$

or

$$A_2 = -\int_c^b f(x)dx$$

So, if you reverse the order of integration, you reverse the sign of the integral.

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

We can now show that

$$\int_a^a f(x)dx = \int_a^b f(x)dx + \int_b^a f(x)dx = \int_a^b f(x)dx - \int_a^b f(x)dx = 0$$

$$m(b-a) \le \int_a^b f(x)dx \le M(b-a)$$

Looking back at the sums we can see that

$$\sum_{i=1}^{n}\left[f(x_i^*) + g(x_i^*)\right]\Delta x_i = \sum_{i=1}^{n} f(x_i^*)\Delta x_i + g(x_i^*)\Delta x_i = \sum_{i=1}^{n} f(x_i^*)\Delta x_i + \sum_{i=1}^{n} g(x_i^*)\Delta x_i$$

Showing that

$$\int_a^b f(x)+g(x)\,dx=\int_a^b f(x)\,dx+\int_b^c g(x)\,dx$$

Substituting $-g(x)$ for $g(x)$ we get

$$\int_a^b f(x)-g(x)\,dx=\int_a^b f(x)\,dx-\int_b^c g(x)\,dx$$

Constants in integrals

$$\sum_{i=1}^{n} cf\left(x_i^*\right)\Delta x_i = c\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x_i$$

so, we have

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$$

Integrals of Absolute Values

$$\int_a^b \left|f(x)\right|dx$$

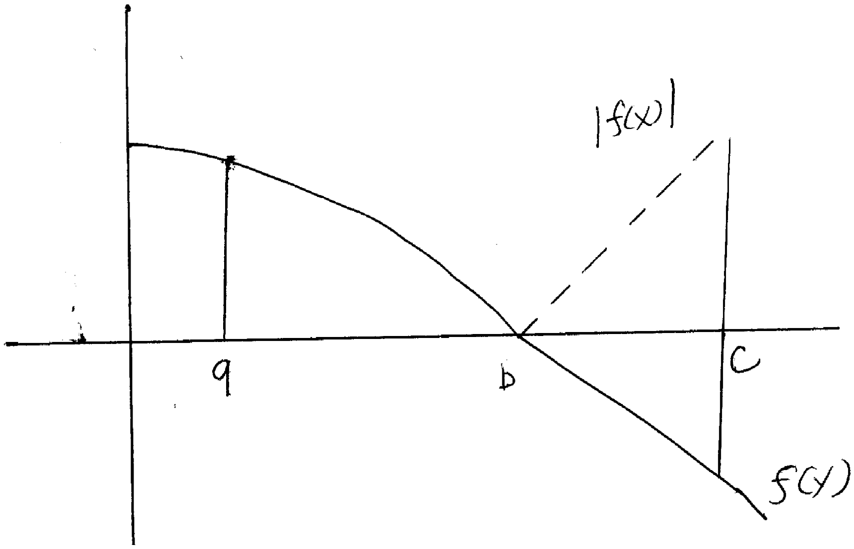
We can divide $[a,c]$ into sub-intervals where $f(x)\geq 0$ or $f(x)\leq 0$.

If $f(x)\geq 0$ on a sub-interval $[a',b']$ then $\int_{a'}^{b'}\left|f(x)\right|dx=\int_{a'}^{b'} f(x)\,dx$

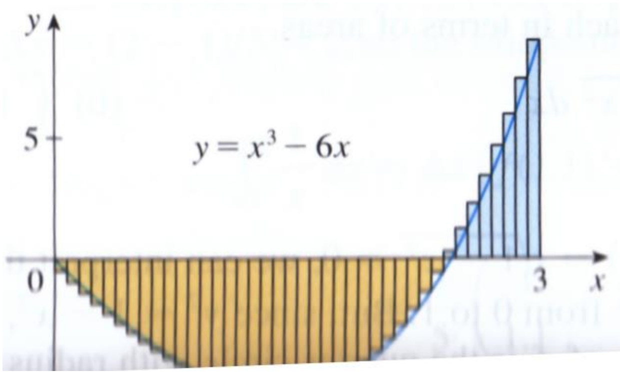
If $f(x)\leq 0$ on a sub-interval $[a',b']$ then $\int_{a'}^{b'}\left|f(x)\right|dx=-\int_{a'}^{b'} f(x)\,dx$

So, we can find the integral $\int_a^b\left|f(x)\right|dx$ by summing the sub-intervals.

Example:



$$\int_a^c |f(x)|dx = \int_a^b |f(x)|dx + \int_b^c |f(x)|dx = \int_a^b f(x)dx - \int_b^c f(x)dx$$



$$x^3 - 6x = 0$$

$$x(x^2 - 6) = 0$$

$$x = \sqrt{6}$$

$$\int_0^3 |x^3 - 6x|\,dx = -\int_0^{\sqrt{6}} x^3 - 6\,dx + \int_{\sqrt{6}}^3 x^3 - 6\,dx$$

$$-\int_0^{\sqrt{6}} x^3 - 6x\,dx = -\left[\frac{x^4}{4} - 3x^2\right]_0^{\sqrt{6}} = -\left[\left(\frac{36}{4} - 18\right) - 0\right] = 9$$

$$\int_{\sqrt{6}}^3 x^3 - 6\,dx = \left[\frac{x^4}{4} - 3x^2\right]_{\sqrt{6}}^3 = \left(\frac{81}{4} - 27\right) - \left(\frac{36}{4} - 18\right) = 2.25$$

$$\int_0^3 |x^3 - 6x|\,dx = 9 + 2.25 = 11.25$$

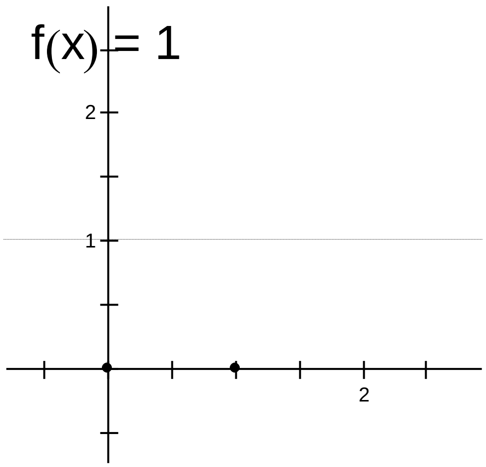
Some Comparison Properties

if $f(x) \geq 0$ for $a \leq x \leq b$ then $\int_a^b f(x)\,dx \geq 0$

if $f(x) \geq g(x)$ for $a \leq x \leq b$ then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

if $m \leq f(x) \leq M$ for $a \leq x \leq b$ then $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$

Note that not all functions are integrable.
For example:



$$f(x) = \begin{cases} 0 & x \in \mathbb{Q} \quad x \text{ is rational} \\ 1 & x \in \mathbb{Q}^c \quad x \text{ is irrational} \end{cases}$$

Condition for Integrability:

If $f(x)$ is a continuous on $[a,b]$ or has at most a finite number of jump discontinuities

then $f(x)$ is integrable on $[a,b]$,

that is $\int_a^b f(x)\,dx$ exists!

Some Examples:

$$\int_1^2 \ln x \, dx$$

$$\int_1^3 x^2 - 2x + 1 \, dx$$

$$\int_1^3 \left| x^2 - 2x + 1 \right| \, dx$$

$$\int_0^\pi \sin x \, dx$$

$$2\int_0^1 \sqrt{1 - x^2} \, dx$$