Math 109 Calc 1 Lecture 3

Calculating Limits

Section 2.3: Finding a limit

As noted earlier, the definition of a limit does not help us calculate a limit. We're going to find some theorems that will assist us in calculating limits. These theorems will start simple and get more complex and more useful. We will prove some, but not all of them.

Recall how we prove a limit.

Given a function $f(x)$, an x value c and limit value l, show that for each small value $\epsilon > 0$ there exists a $\delta > 0$ such that $|x - c| < \delta$ implies that $|f(x) - c| < \epsilon$.

So, our goal is given an ϵ, find a δ. Typically we define δ in terms of ϵ.

Limit of a Constant Function

1. Given $f(x) = K \rightarrow \lim\limits_{x \to c} f(x) = K$

This seems immediately obvious, but the proof is worth reviewing.

Proof: given any $\varepsilon > 0$ we can choose any $\delta > 0$, e.g. $\delta = 1$

So, given $|x - c| < 1$ we have $|f(x) - K| = |0| = 0 < \varepsilon$

Limit of the identity function

2. Given $f(x) = x \rightarrow \lim\limits_{x \to c} f(x) = c$

Proof: Let $\epsilon > 0$ and let $\delta = \epsilon$.

So, we have $|x - c| < \delta = \epsilon$, or $|x - c| < \epsilon$.

But $f(x) = x$ so $|f(x) - c| = |x - c|$. So, by our previous statement $|x - c| < \epsilon$.

Limit of a constant times a function

3. Given $\boldsymbol{\lim\limits_{x \to c} f(x) = m}$ and $\boldsymbol{K}$ a constant then $\boldsymbol{\lim\limits_{x \to c} K\, f(x) = Km}$

Proof:

We wish to show that for $\epsilon > 0$ there is a δ such that $|x - c| < \delta$ implies $|Kf(x) - Km| < \epsilon$.

Let $\epsilon > 0$ so $\frac{\epsilon}{|K|} > 0$. So by assumption there is a δ such that $|x - c| < \delta$ implies $|f(x) - m| < \frac{\epsilon}{|K|}$

Therefore, we have $|K||f(x) - m| < \epsilon$ and $|Kf(x) - Km| < \epsilon$ which means that $\boldsymbol{\lim\limits_{x \to c} K\, f(x) = Km}$

Sum of a Limit

4. If $\lim_{x\to c} \boldsymbol{f(x) = l}$ and $\lim_{x\to c} \boldsymbol{g(x) = m}$ then $\lim_{x\to c} \boldsymbol{f(x) + g(x) = l + m}$

Proof:
Given ϵ > 0, we must show that there exists a δ > 0 such that

$|x-c| < \delta$ implies $[(f(x)+g(x)) - (l+m)] < \epsilon$

Note that $[(f(x)+g(x)) - (l+m)] = [f(x)-l] + [g(x) - m] \leq |f(x)-l|+|g(x) - m|$

For the last step, note that it is always the case that $x \leq |x|$.

We make $[(f(x)+g(x)) - (l+m)] < \epsilon$ by making $[f(x)-l] < \frac{\epsilon}{2}$ and $[g(x) - m] < \frac{\epsilon}{2}$

By our assumption there are values δ_f and δ_g that will accomplish this.

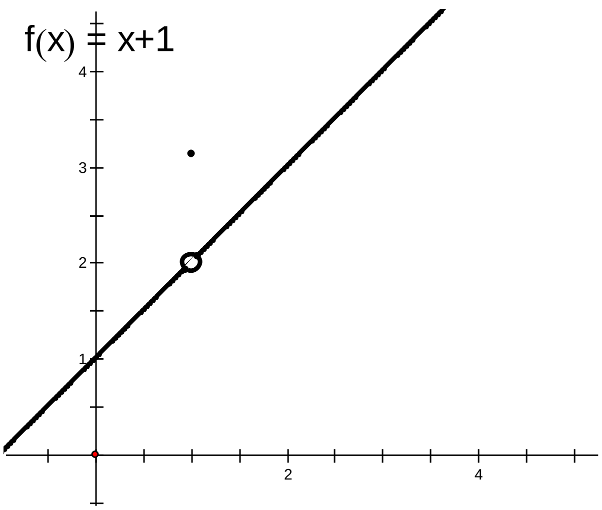
So, letting $\delta = \delta_f + \delta_g$ makes $|f(x)-l| < \frac{\epsilon}{2}$ and $|g(x)-m| < \frac{\epsilon}{2}$.

Finally
$[(f(x)+g(x)) - (l+m)] \leq |f(x)-l|+|g(x) - m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ or

$[(f(x)+g(x)) - (l+m)] < \epsilon$

Example:

$$\lim_{x\to 1} \begin{cases} x + 1, x \neq 1 \\ \pi, x = 1 \end{cases}$$



Since a limit does not depend on the value of the function at the end point, we can rewrite this as

$$\lim_{x\to 1} x + 1 = \lim_{x\to 1} x + \lim_{x\to 1} 1 = 1 + 1 = 2$$

From here on I will not provide ϵ δ proofs, though you can find them in Stewart if you are interested.

Product of Limits

5. If $\boldsymbol{\lim_{x\to c} f(x) = l}$ and $\boldsymbol{\lim_{x\to c} g(x) = m}$ then $\boldsymbol{\lim_{x\to c} f(x)g(x) = lm}$

Power of *n* limit

6. If $\boldsymbol{\lim_{x\to c} f(x) = m}$ and n is an integer then $\boldsymbol{\lim_{x\to c} f(x)^n = m^n}$

This follows from repeated application of 5. The product of limits.

Sum of a Limit

7. If $\boldsymbol{\lim_{x\to c} f(x) = l}$ and $\boldsymbol{\lim_{x\to c} g(x) = m}$ then $\boldsymbol{\lim_{x\to c} f(x) - g(x) = l - m}$

Note how this proceeds directly from 3 and 7 if you set *K*=-1.

Polynomials

8. If ***P*(*x*)** is a **polynomial,** eg**.** $\boldsymbol{P(x) = a_n x^n + a_{n-1}x^{n-1} \cdots a_1 x + a_0}$

then $\boldsymbol{\lim_{x\to c} P(x) = P(c)}$

This follows from 3. Constant times a limit 4. Sum of a limit and 6. The power of a limit.

Examples:

$\lim_{x\to 1}(5x^2 - 12x + 2) = 5(1)^2 - 12(1) + 2 = -5$
$\lim_{x\to 0}(14x^5 - 12x^2 + 2x + 8) = 8$
$\lim_{x\to -1}(2x^3 + x^2 - 2x - 3) = 2(-1)^3 + (-1)^2 - 2(-1) - 3 = -2$
$\lim_{x\to 0}\frac{(x+3)^2-9}{x}$ Here again we can't just plug in because both the numerator and denominator have limits of zero.
$\lim_{x\to 0}\frac{(x+3)^2-9}{x} = \lim_{x\to 0}\frac{x^2+9x+9-9}{x} = \lim_{x\to 0}\frac{x^2+9x}{x} = \lim_{x\to 0} x + 9 = 9$
$\lim_{x\to 0}\frac{\sqrt{x^2+9}-3}{x^2}$ =? This looks particularly tricky, but some basic algebra will help. $\frac{\sqrt{x^2+9}-3}{x^2}\cdot\frac{\sqrt{x^2+9}+3}{\sqrt{x^2+9}+3} = \frac{x^2+9-9}{x^2\left(\sqrt{x^2+9}+3\right)} = \frac{x^2}{x^2\left(\sqrt{x^2+9}+3\right)} = \frac{1}{\sqrt{x^2+9}+3}$ So, $\lim_{x\to 0}\frac{\sqrt{x^2+9}-3}{xx^2} = \lim_{x\to 0}\frac{1}{\sqrt{x^2+9}+3} = \frac{1}{3+3} = \frac{1}{6}$

Reciprocal of a limit

8. If $\lim\limits_{x\to c} f(x) = m$ and $l \neq 0$ *then* $\frac{1}{f(x)} = \frac{1}{m}$

Quotient of a limit

9. If $\lim\limits_{x\to c} f(x) = \mathbf{l}$ and $\lim\limits_{x\to c} g(x) = \mathbf{m}$ with $\mathbf{m} \neq \mathbf{0}$ then $\lim\limits_{x\to c} \frac{f(x)}{g(x)} = \frac{l}{m}$

This follows easily from 5 and 9 the product and reciprocal of a limit.

Note that if If $\lim\limits_{x\to c} f(x) = l$ with $l \neq 0$ and $\lim\limits_{x\to c} g(x) = 0$

then $\lim\limits_{x\to c} \frac{f(x)}{g(x)}$ Does not Exist (DNE)

Examples:

$$\lim_{x\to 2} \frac{3x-5}{x^2+1} = \frac{3(2)-5}{2^2+1} = \frac{1}{5}$$

$$\lim_{x\to -3} \left(x + \frac{1}{x}\right) = \left(-3 + \frac{1}{-3}\right) = -\frac{10}{3}$$

$$\lim_{x\to 3} \frac{x^3-3x^2}{1-x^2} = \frac{27-27}{1-9} = 0$$

$\lim\limits_{x\to 1} \frac{x^2-1}{x-1} =?$ Note that you can't plug in directly because you would get $\frac{0}{0}$

$\frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{x-1} = \left(\frac{x-1}{x-1}\right)(x+1)$ Here the product rule can be used since $\lim\limits_{n\to\infty} \frac{x-1}{x-1} = 1$

so

we have $\lim\limits_{x\to 1} \frac{x^2-1}{x-1} = \left(\lim\limits_{x\to 1} \frac{x-1}{x-1}\right)\left(\lim\limits_{x\to 1} x+1\right) = (1)(2) = 2$

Root of n limit

10. If $\lim\limits_{x\to c} f(x) = m$ and n is an integer then $\lim\limits_{x\to c}\sqrt[n]{f(x)} = \sqrt[n]{m}$

Uniqueness theorem.

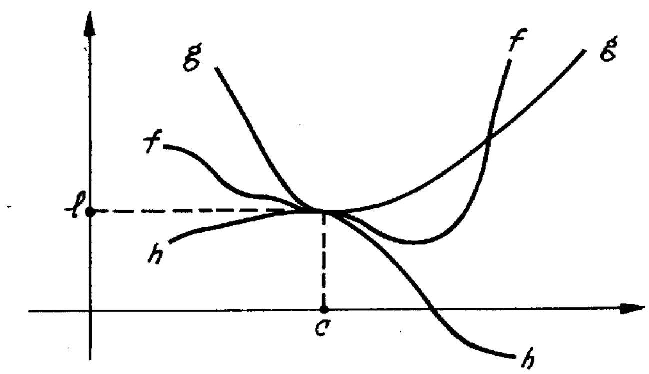
11. If $\lim\limits_{x\to c} f(x) = l$ and $\lim\limits_{x\to c} f(x) = m$ then $l = m$.

This just says that there can only be one limit.

The Pinching or Squeeze Theorem

12. Suppose there is a **number $p > 0$** such that for $0 < |x-c| < p$ that $h(x) \leq f(x) \leq g(x)$ and

$\lim\limits_{x\to c} h(x) = l$ and $\lim\limits_{x\to c} g(x) = l$ then $\lim\limits_{x\to c} f(x) = l$



Examples:

Suppose that $-x^2 \leq f(x) \leq x^2$ for all $x \neq 0$.

Since $\lim\limits_{x\to 0} -x^2 = 0$ and $\lim\limits_{x\to 0} x^2 = 0$ then $\lim\limits_{x\to 0} f(x) = 0$

$\lim\limits_{x\to 0} x^2 sin\frac{1}{x}$ = ?

$-1 \leq sin\frac{1}{x} \leq 1$ so $-x^2 \leq x^2 sin\frac{1}{x} \leq x^2$

We know that $\lim\limits_{x\to 0} x^2 = 0$ and $\lim\limits_{x\to 0} -x^2 = 0$

Therefore $\lim\limits_{x\to 0} x^2 sin\frac{1}{x} = 0$