Math 109 Calc 1 Lecture 29

5.1 Area

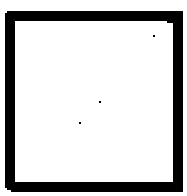
Origins

I imagine that the idea of area became important more than 4000 years ago in the development of agriculture. If you were a farmer and wanted to purchase land for growing, you need to know how to evaluate whether a piece of land that was 100ft x 100ft was more or less than land that was 200ft by 60ft. Of course, the use of the foot as a measure of distance was not used at that time.

Most of what we use today in determining area was known by around 300bc by the Greeks and written down by Euclid.

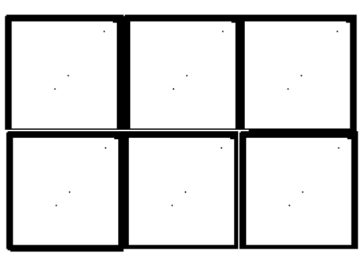
Today we will to investigate the idea of area and how calculus can be used to extend our understanding.

I will start with the simple idea of a square with sides of length 1.



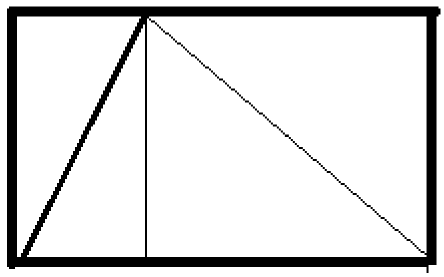
$A_{\square} = s^2$

We call this area 1 square unit.
We expand on this by allowing rectangles of any number of units, with the area being the length times the height.



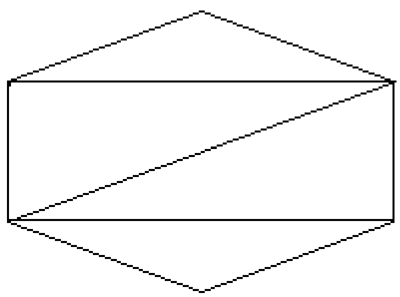
$A_{\square} = lh$

With the use of real numbers, we allow the area of a rectangle of any real number. From here we can develop the idea of the area of a triangle by noticing that a triangle inscribed in a rectangle has half the area.



$A_{\triangle} = \frac{1}{2}bh$

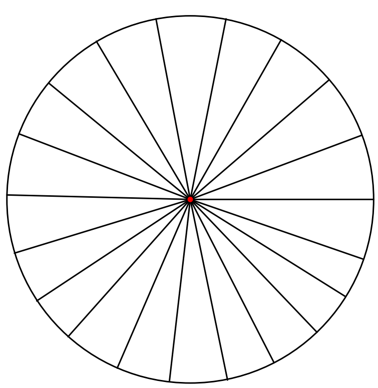
From here we can take any polygon, a multisided object with straight lines and cut it up into triangles, allowing us to assign an area to it.



When looking at the area of objects with non-linear edges there are a few things we know.

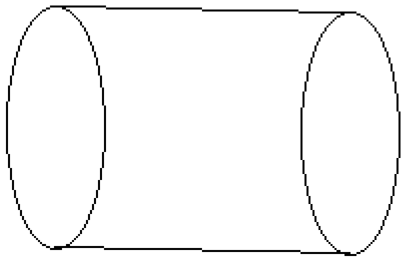
In the case of a circle, we can slice it up into sectors that as we increase the number approximate triangles. With a radius of *r*, the sum of their base is the circumference $c = 2\pi r$, and their height is *r* so the area is found to be

$\frac{1}{2}bh = \frac{1}{2}2\pi r^2 = \pi r^2$

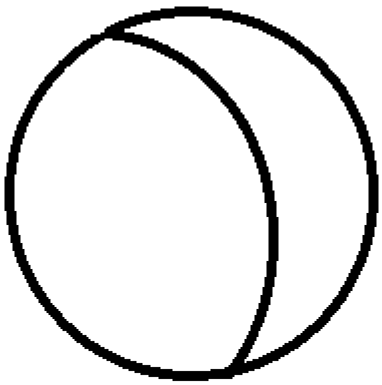


Moving into the realm of 3D objects we can observe that for a cylinder, the outer surface can be unrolled so at give us it’s surface area

$A = 2\pi rh$



Finally, we have the surface area of a sphere which introduces a new complication, the surface is not flat, and cannot be made flat.

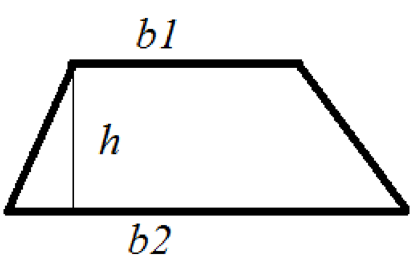


You might remember from high school geometry that the surface area of a sphere is

$A_{\odot} = 4\pi r^2$

You might wonder where this formula comes from.   The Greeks used a technique called the **method of exhaustion**. This method has some similarity to modern day calculus. We will find that calculus provides us a simpler way of finding this area, along with many other odd and complicated shapes.

Before we go forward, a reminder that the area of a trapezoid is $A_{trap} = \frac{(b1+b2)}{2}h$

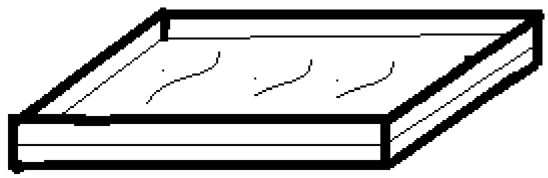


Modern Ideas about Area

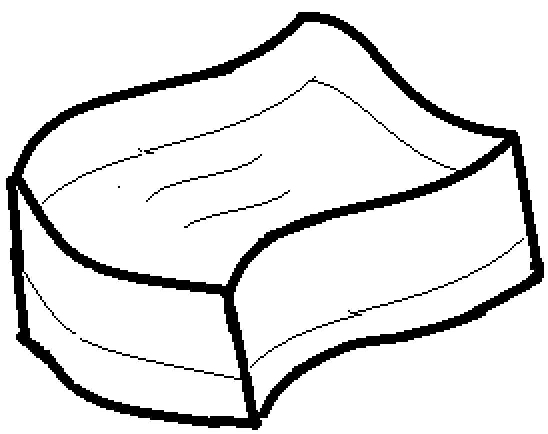
There seems to be a notion among the authors of calculus books that the idea of an area bounded by a curve is a very difficult idea to comprehend.

While this might be true of the precise mathematical definition, I think there are some simple and obvious ways to understand it.

Consider a shallow square dish filled with a liquid to a specific depth:



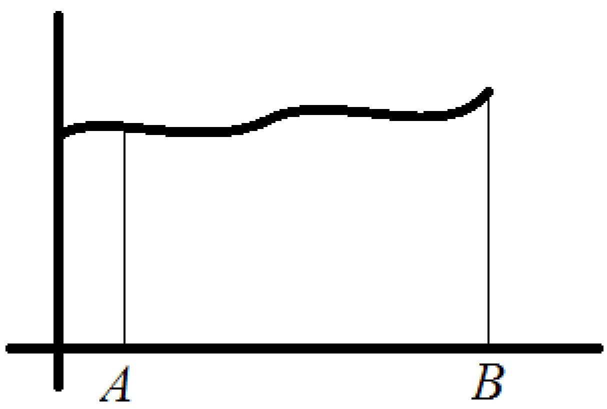
Now let’s say we have a similar dish with a curved boundary that we fill to the same level.



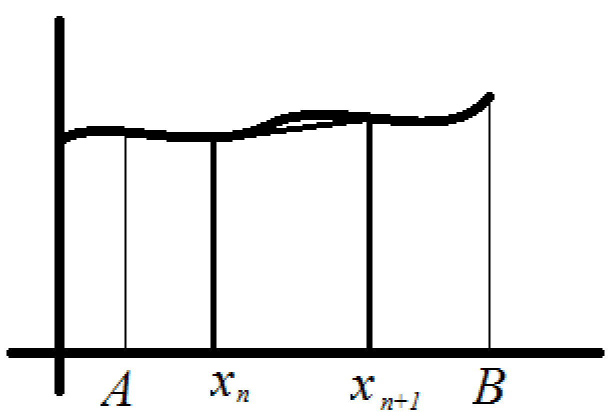
By comparing the volume of the liquids, we could measure the surface area.
This mechanism, or one similar might work for practical purposes, but in the field of mathematics we are interested in a more perfect view of abstract objects.

Calculus and Areas

We are going to start by looking at the problem of finding the area under an arbitrary function.



We break this area up into trapezoids.

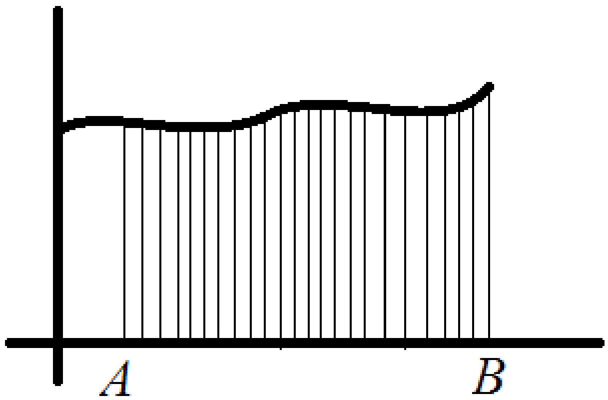


The area of this trapezoid, using the formula we describe earlier is

$$A_{trap} = \frac{\left(f\left(x_{n+1}\right) - f\left(x_n\right)\right)}{2}\left(x_{n+1} - x_n\right)$$

We can label $A = x_0$ and $B = x_M$ so there are M trapezoids.

If we increase the number of trapezoids



We get a better and better approximation of this area.

You can think of this as a limit problem. There's some number that we can get as close to as we want by increasing the number of trapezoids. The sum of these trapezoids if we have M+1 of them is.

$$A=\frac{\left(f\left(x_0\right)+f\left(x_1\right)\right)}{2}\left(x_1-x_0\right)+\frac{\left(f\left(x_1\right)+f\left(x_2\right)\right)}{2}\left(x_2-x_1\right)+\ldots\frac{\left(f\left(x_{M-1}\right)+f\left(x_M\right)\right)}{2}\left(x_M-x_{M-1}\right)$$

If we make the widths all the same, we have $\Delta x=\frac{B-A}{M}$ or $\Delta x=\frac{B-A}{M}$

$$A=\frac{\left(f\left(x_0\right)+f\left(x_1\right)\right)}{2}\Delta x+\frac{\left(f\left(x_1\right)+f\left(x_2\right)\right)}{2}\Delta x+\ldots\frac{\left(f\left(x_{M-1}\right)+f\left(x_M\right)\right)}{2}\Delta x=$$

$$\frac{\Delta x}{2}\left[\left(f\left(x_0\right)+f\left(x_1\right)\right)+\left(f\left(x_1\right)+f\left(x_2\right)\right)+\left(f\left(x_{M-1}\right)+f\left(x_M\right)\right)\right]=$$

$$\frac{(B-A)}{2M}\sum_{n=1}^{M}\left[f\left(x_n\right)+f\left(x_{n+1}\right)\right]$$

Now consider the limit

$$A=\lim_{M\to\infty}\frac{(B-A)}{2M}\sum_{n=1}^{M}f\left(x_n+x_{n+1}\right)$$

This is how we think about the area using calculus, as the limit as M gets very large.

Using a Calculator

It may have already occurred to you that doing the type of calculation described would be fairly easy to implement.

In fact, this is how the Ti-83 calculators work.

The Definite Integral

Doing the calculation on a computer would be a tedious operation.   We will see that calculus provides us with a much easier way to find areas but instead using antiderivatives.

I will introduce some notation here that we will be working with shortly.

$$A = \lim_{M\to\infty} \frac{(B-A)}{2M} \sum_{n=1}^{M} f(x_n) + f(x_{n+1}) = \int_A^B f(x)\,dx$$

This is called a definite integral.

The symbol $\int$ is known as the integral sign. It is meant to look like the letter S to indicate summation.

The trailing *dx* is just notational. This notation comes from Leibnitz. As with the derivative $\frac{dy}{dx}$ you can think of *dx* as a very small number, the Δx that we are summing over, however you should think of it as being infinitesimally small.

The *A* and *B* are the limits between which we are calculating the area.

Note that the *x* is just a dummy variable. You can write the same integral as

$$\int_A^B f(y)\,dy$$

Or

$$\int_A^B f(t)\,dt$$

Or any other letter that is useful without changing the value.

One thing that is good to keep in mind is that a definite integral IS A NUMBER.

If you ever see an expression such as this:

$$\frac{d}{dx}\int_{A}^{B} f(y)\, dy = ?$$

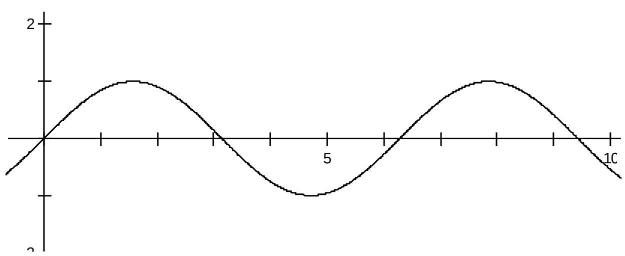
The answer is zero. Since the integral is a number, the function being differentiated is a constant, whose derivative is of course zero.

Some Calculator Examples:

Find the area under the curve $f(x) = x^3 + 2x^2 + 5$ on the interval [0,4] = (100)

Find the area under the curve $f(x) = \sin(x)$ on the interval $[0, 2\pi] = 0$?

Why is that?



Note that between π and 2π the function is below the *x*-axis, so the summation subtracts from the total.

Find the area under the curve e^{-x^2} on the interval [-5,5].

Take your answer and square it. Does the value seem familiar?

This integral is extremely important in statistics. The curve is known as a **Normal** or **Bell curve**.

It describes the distribution of certain types of random events.

Unfortunately, calculus does not provide an easy way of calculating this integral with particular limits, something needed when doing statistics, so the usual procedure is to use a calculator.