Math 109 Calc 1 Lecture 28

4.9 Antiderivatives

Distance/Velocity

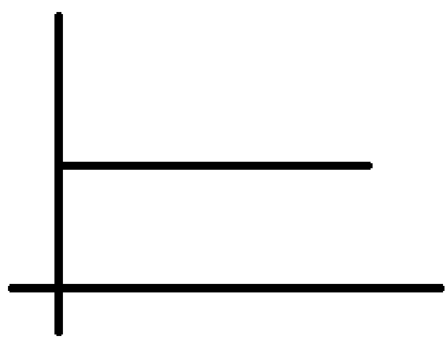
We've talked about how, if you are driving in a car and watching mile markers, you can calculate an average velocity by keeping track of time.

If you keep track over shorter and shorter differences in time the way a speedometer does, you are performing the operation of differentiation.

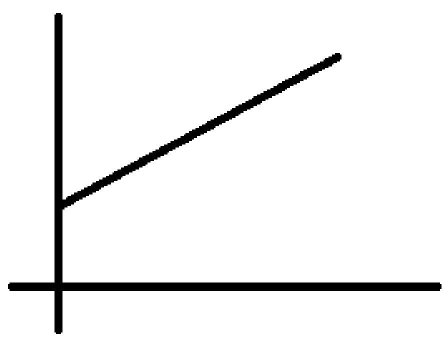
Similarly, if you watch the speedometer and keep track of time, you can calculate how far you've traveled.

That is, from a graph of the velocity, you can draw a graph of the car's position over time. This is the reverse operation of differentiation, finding a function from its derivative.

As a simple example, if your car travels at a constant speed, the velocity graph looks like this.



From which you can construct the distance graph that will look like this.



Note that we can't be sure about where the line starts on the y access without additional informatin.

Antiderivatives

We're going to talk today about antiderivatives. If we have a function which is a derivative and we to find the original function, we are finding the **antiderivative** of that function.

As a simple example,

$$f(x) = x$$

To find the antiderivative we consider the function

$$F(x) = x^2$$

Its derivative is

$$F'(x) = 2x$$

which only differs from what we are looking for by a factor of two. If we change the function $F(x)$ by dividing by 2 we get

$$F(x) = \frac{x^2}{2}$$

and we find that its derivative is what we are looking for

$$F'(x) = x$$

This is a common first strategy for finding antiderivatives.

Look for a function whose derivative might be what we are looking for and adjust any constants that are needed.

As a second example, consider

$$f(x) = \sin(x)$$

The obvious candidate is

$$F(x) = \cos(x)$$

We find its derivative

$$F'(x) = -\sin(x)$$

and we see that we just need to adjust the sign getting

$$F(x) = -\cos(x)$$

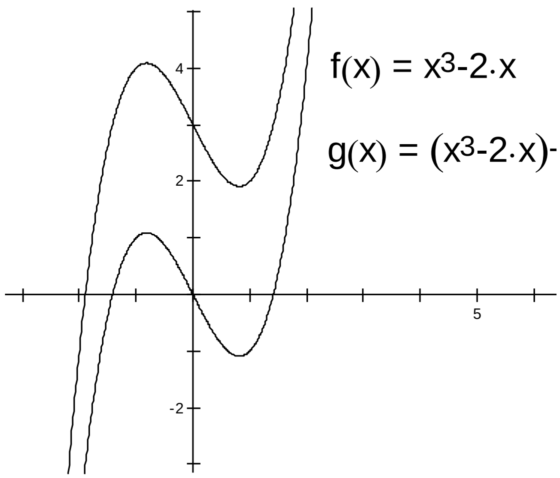
Before we go further, we need to investigate an important issue when finding antiderivatives.

Consider the two functions

$f\left(x\right)=x^3-2x$

$g(x)=x^3-2x+3$

Take a look at their graphs



Now consider their derivatives

$f'\left(x\right)=3x^2-2$

$g'(x)=3x^2-2$

The derivatives are the same, but the functions differ by a constant.
If we are to find the antiderivative of

$f\left(x\right)=3x^2-2$

Which function should we choose?

The answer to this riddle is that when we find an antiderivative, there is always an unknown constant that may appear.

$F\left(x\right)=x^3-2x+C$

Sometimes we have additional information that allows us to find this constant, for example if we know that

$F\left(0\right)=5$

We can see immediately that $C = 5$
We call this a constant of integration

A Digression into General Relativity

You may have heard of Einstein's theory of general relativity. This is his theory of gravity that predicted a number of things that Newton's theory did not.

For one thing, the gravity of large objects can bend light.
Another thing the theory predicts is black holes, which we now know do exist.

The main equation of this theory can be written in a very elegant way.

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}$$

This uses a very compact notation hiding the fact that it represents 256 equations.
Many of these equations are duplicates due to symmetry, so there are only 10 independent equations.

The original version of the equations written down by Einstein excluded the middle term:

$$G_{\mu\nu} = \kappa T_{\mu\nu}$$

The symbol Λ in the equation is known as the **Cosmological Constant**.

The cosmological constant is a constant of integration.

After adding this constant into his equations Einstein decided it was a mistake which he called his "greatest blunder".

Decades later in 1998, it was discovered that the expansion of the universe is accelerating.
This is described by the cosmological constant having a positive number.
We don't know why it has this value.
One theory is that space has a repulsive energy, which we call **Dark Energy**.

In any case, this is a good example indicating that we should not ignore constants of integration.

The Antiderivative of Polynomials

Let's look at the antiderivative of $f(x) = x^n$

We suspect that it will be related to $F(x) = x^{n+1}$

By differentiating $F(x)$ we get

$$F'(x) = (n+1)x^n$$

So, we can adjust this by setting

$$F(x) = \frac{x^{n+1}}{n+1}$$

From this we can see that the antiderivative of

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

is

$$F(x) = \frac{a_n x^{n+1}}{n+1} + \frac{a_{n-1} x^n}{n} + \ldots + \frac{a_1 x^2}{2} + a_0 x + C$$

A Topiary Garden of Antiderivatives

At this point it might be a good idea to list some of the antiderivatives that we can find from our prior knowledge of derivatives

$f(x)$	$F(x)$
$x^n \quad \to \quad n \neq -1$	$\dfrac{x^{n+1}}{n+1}+C$
e^x	$e^x + C$
$\cos(x)$	$\sin(x)+C$
$\sin(x)$	$-\cos(x)+C$
$\sec^2(x)$	$\tan(x)+C$
$\sec(x)\tan(x)$	$\sec(x)+C$
$\dfrac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x + C$
$\dfrac{1}{1+x^2}$	$\tan^{-1} x + C$
$\cosh(x)$	$\sinh(x)+C$
$\sinh(x)$	$\cosh(x)+C$

This brings us to the question of the anti-derivative of $\frac{1}{x}$?

We know that for $x > 0$ $(\ln x)' = \frac{1}{x}$

We also know that for $x < 0$ $(\ln - x)' = \frac{1}{x}$

So, in general we have the antiderivative of $\frac{1}{x}$ is $\ln|x|$

If you know that $x>0$ you can leave off the absolute value.

Some Antiderivative Rules

Constants:

Since $\frac{d}{dx}cf(x) = cf\,'(x)$ we can conclude that the antiderivative of $cf\,'(x)$ is $cf(x)+C$

Sums:

Since $\frac{d}{dx}\left(f(x)+g(x)\right) = f\,'(x)+g\,'(x)$ we can conclude that the antiderivative of $f\,'(x)+g\,'(x)$ is

$f(x)+g(x)+C$

We will be studying other ways to find antiderivatives, however, now is a nice time to introduce some patterns that will occur quite often and are worth looking for.

Since $\left[f(x)^n\right]' = nf(x)^{n-1}f\,'(x)$ if you see a function of the form $f(x)^{n-1}f\,'(x)$ you can find it's antiderivative $\frac{f(x)^n}{n}$

Example:

Find the antiderivative of $\sin(x)\cos(x)$.

Let $f(x)=\sin(x)$, $f\,'(x)=\cos(x)$ and $n-1=1$, or n=2, so we have the antiderivative $F(x)=\frac{\sin^2(x)}{2}+C$

Since $\frac{d}{dx}\ln|f(x)| = \frac{f'(x)}{f(x)}$

the antiderivative of $\frac{f'(x)}{f(x)}$ is $\frac{d}{dx}\ln|f(x)| + C$

Example:

What is the antiderivative of $\frac{x}{x^2+1}$?

Since $\frac{d}{dx}x^2 = 2x$ we rewrite our problem as $\frac{1}{2}\frac{2x}{x^2+1}$ and we conclude that the antiderivative is $\frac{1}{2}\ln\left|x^2+1\right| = \ln\sqrt{x^2+1} + C$

Example:

Find the antiderivative of

$(x-1)(x+2)$

First multiply using FOIL

$x^2 + x - 2$

Then use the reverse power rule to get

$F(x) = \frac{x^3}{3} + \frac{x^2}{2} - 2x + C$

Example:

Find the antiderivative of

$$\sqrt{x-1} = (x-1)^{\frac{1}{2}} = \frac{2}{3} \cdot \frac{3}{2}(x-1)^{\frac{1}{2}}$$

Notice that $\sqrt{x-1} = (x-1)^{\frac{1}{2}} = \frac{2}{3} \cdot \frac{3}{2}(x-1)^{\frac{1}{2}}$

So

$$F(x) = \frac{2}{3}(x-1)^{\frac{3}{2}} + C$$

Example:

Find the antiderivative of

$$\frac{1}{\sqrt{x}}$$

Notice that $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$

So

$$F(x) = 2\sqrt{x} + C$$

Example:

Find the antiderivative of

$$x\left(x^2-1\right)^4$$

We could multiply this out into a standard polynomial form but notice that $\frac{d}{dx}\left(x^2-1\right) = 2x$

So

$$F(x) = \frac{1}{2 \cdot 5}\left(x^2-1\right)^5 + C$$

Example:

Find the antiderivative of

$$e^x + \frac{20}{1+x^2} \text{ with } F(0) = -2$$

By now this should be straight forward

$$F(x) = e^x + 20\tan^{-1} x + C$$

$$F(0) = e^0 + 20\tan^{-1} 0 + C = -2$$
$$C = -2 - 1 = -3$$

So, the particular solution is

$$F(x) = e^x + 20\tan^{-1} x - 3$$

Example:

Given that

$$f''(x) = 12x^2 + 6x - 4 \text{ find } f \text{ given that } f(0) = 4, \quad and \quad f(1) = 1$$

We have

$$f'(x) = 4x^3 + 3x^2 - 4x + C$$
$$f(x) = x^4 + x^3 - 2x^2 + Cx + D$$
$$f(0) = D = 4$$
$$f(1) = 1 + 1 - 2 + C + 4 = 1$$
$$C = -3$$
$$f(x) = x^4 + x^3 - 2x^2 + -3x + 4$$