Math 109 Calc 1 Lecture 26

Optimization

When we talk about **optimization**, generally we are talking about maximizing or minimizing something.

In a particular situation, one might have choices about what one optimizes. For example, as students you might want to optimize:

1. Your free time
2. The amount you learn
3. Your GPA
4. Your future income

Note that these can be in conflict, maximizing your free time and the amount you learn seem to be. Of course, there might be some compromise that itself you want to optimize.

For example, you might want to try to maximize your happiness. Learning can bring you happiness as well as free time, but too much of each will not. The same can be said for future income.

There are various strategies one might use to optimize. Mathematics is not always the easiest or best. One possibility is to build a physical model and do experiments on it.

This brings us back to soap bubbles and wire frames. For many complicated surfaces, it's easier to build a wire frame and observe what soap bubbles do than to calculate the minimum surface:

3 min video:
https://www.youtube.com/watch?v=hNxpp4ufdGA

Optimization sometimes appears in scientific laws and theories.

Preparing for a Revolution 109

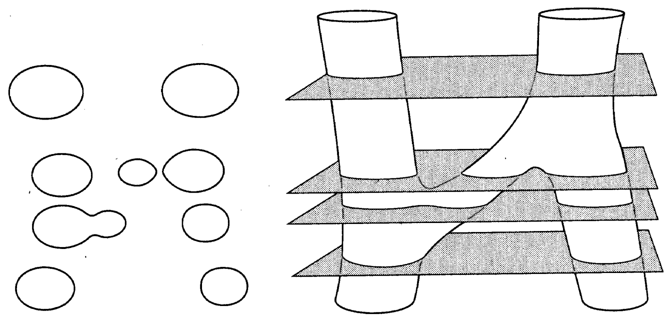


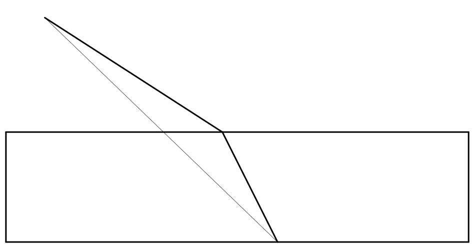
Fig. 6. The propagation and interaction of strings are determined by the same law, which is to minimize the area of the surface in spacetime. On the right, we see the surface in spacetime traced out by two closed strings, which interact by exchanging a third closed string. On the left, we see the sequence of configurations in space, which come from taking slices through the picture in spacetime on the right. First we see two closed strings, then one splits off a third closed string, which travels and then joins the second string.

Ref: “The Trouble with Physics”, by Lee Smolin, p. 109.

In String theory, all fundamental particles are described as strings which propagate through time.

Their interactions of these strings are governed by the minimization of the space-time surface.

In optics, when trying to find the index of refraction of a material such as glass, the speed of light in a transparent material will slow down. It’s a curious principle that the path that light takes through a material minimizes the time taken.



So, we can calculate the index of refraction by knowing the speed of light through the material.

Linear Optimization

You may have already seen linear optimization or linear programming in another math course.

Here's an example:

Let's say we have some quantity that is modeled by the equation

$$z = 2x + 3y$$

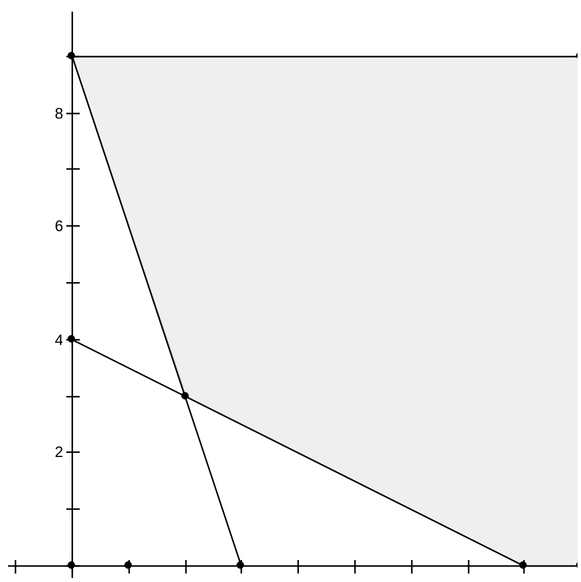
that we want to minimize, subject to some constraints.

z might be the cost of something we want to build, and *x* and *y* might be the cost of some of the parts we need.

The constraints we will consider are the following.

$$x \geq 0$$
$$y \geq 0$$
$$3x + 6y \geq 24$$
$$y \geq -3x + 9$$

The strategy we would use to solve this would be to first graph the inequalities to see where the "feasible region" is.



A theorem tells us that z will be minimized or maximized at the intersection of the graphed lines, the points (0,8), (2,3) and (8,0)

At these points we find that

$$\left(0,9\right) \quad z = 2(0) + 3(9) = 27$$
$$\left(2,3\right) \quad z = 2(2) + 3(3) = 13$$
$$\left(8,0\right) \quad z = 2(8) + 3(0) = 16$$

So, we can see that for x=2 and y=3 we have a minimum of z=13.

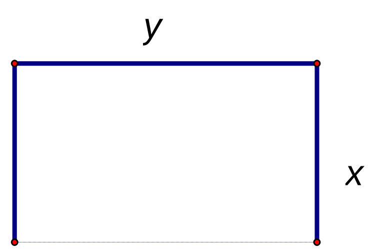
Non-linear Optimization

For problems that are non-linear, we now have tools from calculus that allow us to optimize. If we can express the variable we want to optimize in terms of a function, we can use ability to find the maximum and/or minimum of that function to solve the problem.

You will find that most of these problems require a minimum recollection of geometry.

Example:

We have 2400 feet of fencing material.
We want to create a 3-sided fence. The fourth side could be a river or a wall. We want to maximize the enclosed area.



We have the constraint that $2x + y = 2400$ and we want to maximize $A = xy$.

First, we want to express *z* as a function of one variable, so we solve the first equation for *y*.

$$y = 2400 - 2x$$

and then we substitute *y* in the second equation.

$$A = x(2400 - 2x) = -2x^2 + 2400$$

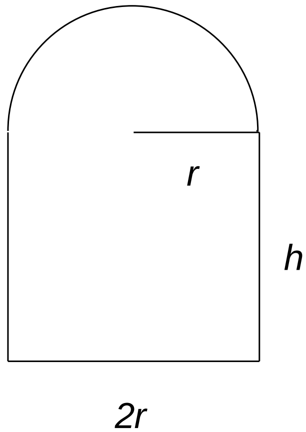
To minimize *A* we find $\frac{dA}{dx} = -4x + 2400$, and set it equal to zero

$$-4x + 2400 = 0$$
$$x = 600$$

At *x*=600 we find that *z* = 1200 and *A*=7200sq ft.

Example:

A window in the shape of a rectangle topped by a semi-circle is to be surrounded by a metal border of length 16 feet. What is the radius of the circle if the area of the window is to be maximized?



We have

The area of the semi-circle is $\frac{\pi r^2}{2}$

The area of the rectangle is $2r \cdot h$

So, the area of the window is

The circumference of the window is $C = 2r + 2h + \pi r = 16$

Solving this last equation for $2h$ we get $2h = 16 - 2r - \pi r = 16 - (2+\pi)r$.

Substituting into the area equation we get

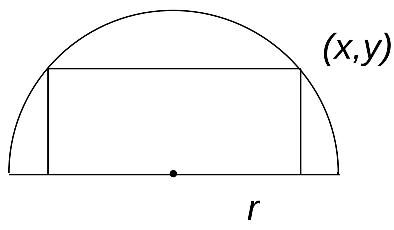
$$A(r) = \frac{\pi r^2}{2} + 16r - (2+\pi)r^2 =$$
$$16r - \frac{(4+\pi)}{2}r^2$$

Finding the derivative and setting it to zero

$$\frac{dA}{dr} = 16 - (4+\pi)r = 0$$
$$r = \frac{16}{4+\pi}$$

Example

Find the area of the largest rectangle that can be inscribed in a semi-circle.



Given the equation of this circle, $x^2 + y^2 = r^2$, we see that the points on the circle have coordinates

$$\left(\pm x, \sqrt{r^2 - x^2}\right)$$

So, the area of the rectangle is

$$A(x) = hw = 2x\sqrt{r^2 - x^2}$$

Finding the derivative and setting it to zero we get

$$\frac{dA}{dx} = 2\left[x \cdot \frac{-2x}{2\sqrt{r^2 - x^2}} + \sqrt{r^2 - x^2}\right] = 2\left[\sqrt{r^2 - x^2} - \frac{x^2}{\sqrt{r^2 - x^2}}\right] = \frac{2\left(r^2 - 2x^2\right)}{\sqrt{r^2 - x^2}} = 0$$

Since $r^2 > x^2$ we can write this as

$$2\left(r^2 - 2x^2\right) = 0$$

$$r^2 = 2x^2$$

$$x = \pm\frac{1}{\sqrt{2}} r$$

Plugging this into our area equation we get

$$A\left(\frac{r}{\sqrt{2}}\right) = 2\frac{r}{\sqrt{2}}\sqrt{r^2 - \left(\frac{r}{\sqrt{2}}\right)^2} = 2\frac{r}{\sqrt{2}}\sqrt{r^2 - \frac{r^2}{2}} = 2\frac{r}{\sqrt{2}}\sqrt{\frac{r^2}{2}} = r^2$$

In Business, there are many situations where one wants to optimize.   A typical problem is as follows.

Example:

A game console manufacturer determines that in order to sell x units, the price per one unit (in dollars) must decrease by the linear law (the demand function)

$$p(x) = 500 - 0.1x \cdot \left(\frac{\$}{device} \right)$$

The manufacturer also determines that the cost depends on the volume of production and includes a fixed part 100,000($) and a variable part 100x, that is C(x)=100000+100 x.

What price per unit must be charged to get the maximum profit?

The total revenue is given by

R(x)= xp(x) = x(500−0.1x) = 500x−0.1x2

The profit is determined by the formula

$$P(x) = R(x) - C(x) = 500x - 0.1x^2 - (100000 + 100x) = 400x - .1x^2 - 100000$$

Find the derivative of P(x):

$$P'(x) = 400 - 0.2x$$

There is one critical value:

$$P'(x) = 0 \quad \Rightarrow \quad 400 - 0.2x = 0 \quad \Rightarrow \quad x = 2000$$

We use the Second Derivative Test to classify the critical point.

$$P''(x) = -0.2 < 0$$

Since P′′(x) is negative, x=2000 is a point of maximum.

Hence, the profit is maximized when 2000 game consoles are sold.

In this case, the price per unit is equal to

$$p(x = 2000) = 500 - 0.1 \cdot 200 = 300 device \left(\frac{\$}{device} \right)$$

Example:

The production cost per a period of time is given by the quadratic function

$$C(x) = a + bx^2$$

where a,b are some positive real numbers and x represents the number of units. Find the minimal average cost. (The average cost is the total cost divided by the number of units produced.)

By definition, the average cost $\overline{C}$ is written in the form

$$\overline{C}(x) = \frac{C(x)}{x} = \frac{a+bx^2}{x} = \frac{a}{x} + bx$$

Take the derivative and set it equal to zero to find the critical points:

$$\overline{C}'(x) = \frac{-a}{x^2} + b = \frac{bx^2 - a}{x^2}$$

$$\overline{C}'(x) = 0, \quad \Rightarrow \quad \frac{bx^2 - a}{x^2} = 0 \quad \Rightarrow \quad bx^2 - a \quad \Rightarrow \quad x = \sqrt{\frac{a}{b}}$$

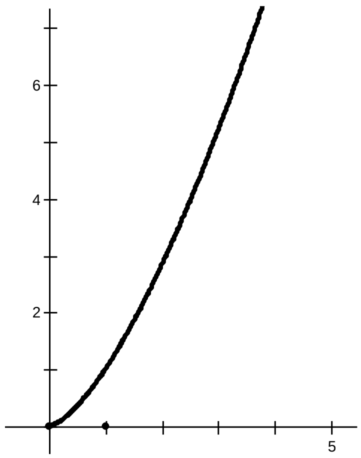
By the first derivative test, we identify that this point is a point of minimum.

Calculate the minimal average cost:

$$\overline{C}_{\min} = \overline{C}\left(\sqrt{\frac{a}{b}}\right) = \frac{a}{\sqrt{\frac{a}{b}}} + b\sqrt{\frac{a}{b}} = \sqrt{ab} + \sqrt{ab} = 2\sqrt{ab}$$

Example
A distance problem

Find the point on the curve $y = x^{3/2}$ which is closest to the point $\left(\frac{1}{2}, 0\right)$.



Note that this function is not defined for $x < 0$

We use the distance formula between the points $\left(x, x^{3/2}\right)$ and $\left(\frac{1}{2}, 0\right)$.

$$D(x) = \sqrt{\left(\frac{1}{2} - x^{3/2}\right)^2 + \left(0 - x\right)^2}$$

Since it will give us the same answer, we can find the derivative of $D(x)^2$ more easily.

$$D(x)^2 = \left(\frac{1}{2} - x\right)^2 + \left(x^{3/2}\right)^2 = x^3 + x^2 - x + \frac{1}{4}$$

$$\left[D(x)^2\right]' = 3x^2 - 2x - 1 = 0$$

$$\frac{-2 \pm \sqrt{4+12}}{6} = \frac{-2 \pm 4}{6} = -1, \frac{1}{3}$$

The point -1 is outside the domain, but we need to consider x=0.

$$D(0) = D(x) = \frac{1}{2}$$

$$D\left(\frac{1}{3}\right) = \sqrt{\left(\frac{1}{3}\right)^3 + \left(\frac{1}{3}\right)^2 - \frac{1}{3} + \frac{1}{4}} = \sqrt{\frac{1}{27} + \frac{1}{9} - \frac{1}{3} + \frac{1}{4}} = \sqrt{\frac{3\left(4 + 12 - 36 + 27\right)}{324}} =$$

$$\frac{\sqrt{3 \cdot 7}}{18} = \frac{\sqrt{21}}{18} < \frac{1}{2}$$