Math 109 Calc 1 Lecture 23

Indeterminant Forms, L'Hospital's Rule

Section 4.4

Looking back at Limits

Recall that one definition of a derivative that we've used is

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

An equivalent definition for the derivative at a point *a* is

$$f'(x) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

Consider two continuous functions $f(x)$ and $g(x)$ where
$f(a)=g(a)=0$.
Also assume that $g'(a)\neq 0$ and consider the following limit

$$\lim_{x\to a}\frac{f'(x)}{g'(x)} = \frac{\lim_{x\to a}\dfrac{f(x)-f(a)}{x-a}}{\lim_{x\to a}\dfrac{g(x)-g(a)}{x-a}} = \lim_{x\to a}\frac{\dfrac{f(x)-f(a)}{x-a}}{\dfrac{g(x)-g(a)}{x-a}} =$$

$$\lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)}\cdot\lim_{x\to a}\frac{x-a}{x-a} = \lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)}$$

Since $f(a)=g(a)=0$ we have $\lim_{x\to a}\frac{f'(x)}{g'(x)} = \lim_{x\to a}\frac{f(x)}{g(x)}$

Note that in this last expression, both the numerator and denominator have limits of zero, giving rise to the term an **indeterminant form**, in this case $\frac{0}{0}$.

This is a simplified version of what is called L'Hospital's rule. It also applies to one sided limits as well as limits at infinity. We will see that it applies to a number of indeterminant forms.

This name comes from the Marquis de L'Hospital who did not discover this. Instead it was discovered by Johann Bernoulli, a member of a large family of famous and productive mathematicians and physicists. The Marquis purchased the formula and then published it as his own. Such exchanges are frowned upon in academia today.

Example:

A simple example would be $\lim_{x \to 1} \frac{\ln x}{x-1}$

Here we see that the requirements are fulfilled.
If x = 1 then $\ln(x) = x - 1 = 0$.
Also, the derivative of the denominator, $g'(x) = 1 \neq 0$

Using L'Hospital's rule we find

$$\lim_{x \to 1} \frac{\ln x}{x-1} = \lim_{x \to 1} \frac{(\ln x)'}{(x-1)'} = \lim_{x \to 1} \frac{\frac{1}{x}}{1} = 1$$

L'Hospital's rule is somewhat more general in that it can be applied to a number of different types of **indeterminate forms**.

An indeterminate form is the where the result of a limit cannot be determined by merely using the continuity of functions to determine the final result.

Examples of Indeterminate forms:

$$\frac{0}{0}$$

This is a common form where we have a limit in which the numerator and the denominator both approach zero. The result is indeterminate and may be zero, a non-zero value or ∞ depending on the behavior of the two functions.

$$\frac{\pm\infty}{\infty}$$

This form occurs when both the numerator and the denominator of a function both increase without bound or decrease without bound.

Example:

$$\lim_{x\to\infty}\frac{x^2-1}{2x^2+1}$$

It's possible to evaluate this without L'Hospital's rule by dividing both the numerator and the denominator by x^2

$$\lim_{x\to\infty}\frac{1-\frac{1}{x^2}}{2+\frac{1}{x^2}}$$

Since $\lim_{x\to\infty}\frac{1}{x^2}=0$ we have

$$\lim_{x\to\infty}\frac{x^2-1}{2x^2+1}=\frac{1}{2}$$

Using L'Hospital's rule we find

$$\lim_{x\to\infty}\frac{x^2-1}{2x^2+1}=\lim_{x\to\infty}\frac{(x^2-1)'}{(2x^2+1)'}=\lim_{x\to\infty}\frac{2x}{4x}=\lim_{x\to\infty}\frac{2}{4}\cdot\lim_{x\to\infty}\frac{x}{x}=\frac{1}{2}$$

It's **EXTREMELY** important to verify that a limit results in an indeterminate form before applying L'Hospital's rule. Otherwise you can get an incorrect result.

Example:

$$\lim_{x\to 0}\frac{x+5}{2x+5}$$

Clearly the limit here is zero, however if one were to incorrectly apply L'Hospital's rule one would get

$$\lim_{x\to 0}\frac{x+5}{2x+5}=\lim_{x\to 0}\frac{(x+5)'}{(2x+5)'}=\lim_{x\to 0}\frac{1}{2}=\frac{1}{2}$$

Example:

$$\lim_{x\to\infty}\frac{e^x}{x^2}$$

First we note that $\lim_{x\to\infty} e^x = \infty$ and $\lim_{x\to\infty} x^2 = \infty$ so we can apply L'Hospital.

$$\lim_{x\to\infty}\frac{e^x}{x^2} = \lim_{x\to\infty}\frac{e^x}{2x}$$

Here we are again faced with the indeterminant form $\frac{\infty}{\infty}$ so we can apply the rule again.

$$\lim_{x\to\infty}\frac{e^x}{2x} = \lim_{x\to\infty}\frac{e^x}{2} = \infty$$

Other Indeterminate Forms

There are other indeterminate forms for which L'Hospital's rule can be applied.

$0\cdot\infty$

$\infty-\infty$

∞^0

0^∞

1^∞

0^0

We shall try some examples.

$0 \cdot \infty$

Example:

$$\lim_{x \to 0^+} x \ln(x)$$

We can rewrite this as $\frac{\infty}{\infty}$ as follows

$$\lim_{x \to 0^+} x \ln(x) = \lim_{x \to 0^+} \frac{\ln x}{1/x} = \lim_{x \to 0^+} \frac{(\ln x)'}{(1/x)'} = \lim_{x \to 0^+} \frac{1/x}{-(1/x^2)} = \lim_{x \to 0^+} -x = 0$$

$\infty - \infty$

Example:

$$\lim_{x\to 1^+}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)$$

Both of these terms increase without bound so it is unclear what will happen to the limit. Once again we modify the expression, this time putting the terms over a common denominator.

$$\lim_{x\to 1^+}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)=\lim_{x\to 1^+}\left(\frac{x-1-\ln x}{(\ln x)(x-1)}\right)=\frac{0}{0}$$

Since both the numerator and denominator go to zero we can apply L'Hopital's rule.

$$\lim_{x\to 1^+}\left(\frac{x-1-\ln x}{(x-1)\ln x}\right)=\lim_{x\to 1^+}\frac{(x-1-\ln x)'}{((x-1)\ln x)'}=\lim_{x\to 1^+}\frac{1-\frac{1}{x}}{(x-1)\cdot\frac{1}{x}+\ln x}=\lim_{x\to 1^+}\frac{1-\frac{1}{x}}{1-\frac{1}{x}+\ln x}$$

Multiplying the top and bottom by x we get $\lim_{x\to 1^+}\frac{x-1}{x-1+x\ln x}=\frac{0}{0}$

Since this also goes to the indeterminant form $\frac{0}{0}$ we can apply the rule again.

$$\lim_{x\to 1^+}\frac{x-1}{x-1+x\ln x}=\lim_{x\to 1^+}\frac{(x-1)'}{(x-1+x\ln x)'}=\lim_{x\to 1^+}\frac{1}{1+x\cdot\frac{1}{x}+\ln x}=\lim_{x\to 1^+}\frac{1}{2+\ln x}=\frac{1}{2}$$

∞^0

Example:

$$\lim_{x\to 0}\left(\frac{1}{x}\right)^x$$

First, we note that if $y=\lim_{x\to 0}\left(\frac{1}{x}\right)^x$ then

$$\ln y=\lim_{x\to 0}\ln\left(\frac{1}{x}\right)^x=\lim_{x\to 0}-x\ln x=\lim_{x\to 0}\frac{-\ln x}{\frac{1}{x}}=\frac{-\infty}{\infty}$$

This allows us to apply the rule

$$\ln y=\lim_{x\to 0}\frac{-\ln x}{\frac{1}{x}}=\lim_{x\to 0}\frac{(-\ln x)'}{\left(\frac{1}{x}\right)'}=\lim_{x\to 0}\frac{-\frac{1}{x}}{-\frac{1}{x^2}}=\lim_{x\to 0}x=0$$

Of course, this means that $y=\lim_{x\to 0}\left(\frac{1}{x}\right)^x=1$

1^{∞}

Example:

$$\lim_{x\to 1} x^{1/(1-x)}$$

We again start with

$$y = \lim_{x\to 1} x^{1/(1-x)}$$

$$\ln y = \lim_{x\to 1} \ln x^{1/(1-x)} = \lim_{x\to 1} \frac{\ln x}{1-x} = \lim_{x\to 1} \frac{(\ln x)'}{(1-x)'} = \lim_{x\to 1} \frac{\frac{1}{x}}{-1} = -1$$

$$y = \frac{1}{e}$$

0^0

Example:

$$\lim_{x\to 0} x^x$$

If $y = \lim_{x\to 0} x^x$ we have

$$\ln y = \lim_{x\to 0} \ln x^x = \lim_{x\to 0} x \ln x = \lim_{x\to 0} \frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0} \frac{(\ln x)'}{\left(\frac{1}{x}\right)'} = \lim_{x\to 0} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x\to 0} -x = 0$$

So, $y = 1$

Sometimes it requres multiple applications of the rule

Example:

$$\lim_{x\to 0}\frac{\tan x - x}{x^3} =$$

1) $\lim_{x\to 0}\frac{\tan x - x}{x^3} = \lim_{x\to 0}\frac{\sec^2 x - 1}{2x^2}$

2) $\lim_{x\to 0}\frac{\sec^2 x - 1}{2x^2} = \lim_{x\to 0}\frac{2\sec^2 x\tan x}{6x}$

$$\lim_{x\to 0}\frac{2\sec^2 x\tan x}{6x} = \left(\frac{1}{3}\lim_{x\to 0}\sec^2 x\right)\left(\lim_{x\to 0}\frac{\tan x}{x}\right)$$

$$\left(\frac{1}{3}\lim_{x\to 0}\sec^2 x\right)\left(\lim_{x\to 0}\frac{\tan x}{x}\right) = \frac{1}{3}\lim_{x\to 0}\frac{\tan x}{x}$$

3) $\frac{1}{3}\lim_{x\to 0}\frac{\tan x}{x} = \frac{1}{3}\lim_{x\to 0}\frac{\sec^2 x}{1} = \frac{1}{3}$

$$\lim_{x\to 0}\frac{\tan x - x}{x^3} = \frac{1}{3}$$