Math 109 Calc 1 Lecture 22

Derivatives and the Shape of Graphs (More)

Section 4.3

Increasing and Decreasing

Recall how we can tell if a function is increasing or decreasing.

If $f'(x) > 0$ on an interval, the function is increasing on the interval.
If $f'(x) < 0$ on an interval, the function is decreasing on the interval.

The first Derivative Test

If *c* is a critical point and the derivative changes from positive to negative, then the function has a maximum at *c*.

If *c* is a critical point and the derivative changes from negative to positive, then the function has a minimum at *c*.

If *c* is a critical point and the derivative changes from negative to negative or positive to positive, then the function has not maximum or minimum.

If $f''(x) > 0$ on an interval, the function is concave up
If $f''(x) < 0$ on an interval, the function is concave down

The Second Derivative Test

If a function *f* has a second derivative that is continuous near *c*,

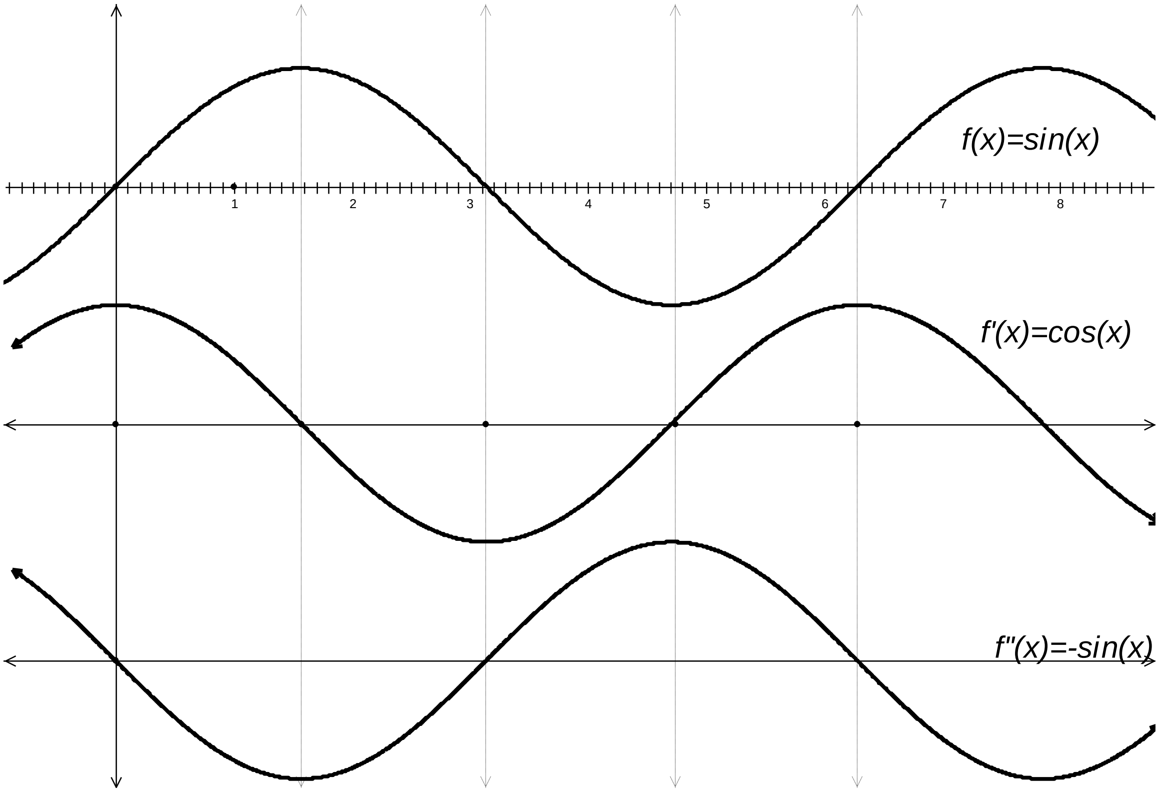
If $f'(c) = 0 \; and \; f''(c) > 0$ then *f* has a local minimum at *c*.

If $f'(c) = 0 \; and \; f''(c) < 0$ then *f* has a local maximum at *c*.

Points of inflection
If $f''(c) = 0$ then c can be a point of inflection where the concavity changes direction.

Example:

$f(x) = \sin(x) \quad f'(x) = \cos(x) \quad f''(x) = -\sin(x)$



Note that on $\left[0, \frac{\pi}{2}\right]$ $f(x) = \sin(x)$ is increasing and $f'(x) = \cos(x)$ is positive, while on $\left[\frac{\pi}{2}, \pi\right]$ $f(x) = \sin(x)$ is decreasing and $f'(x) = \cos(x)$ is negative.

On the $[0, \pi]$ we see that $f''(x) = -\sin(x) < 0$ indicating that $\sin(x)$ is concave down.
On the $[\pi, 2\pi]$ we see that $f''(x) = -\sin(x) > 0$ indicating that $\sin(x)$ is concave down.

There is a critical point on $f(x) = \sin(x)$ at $\frac{\pi}{2}$ where $\cos(x) = 0$. Indicating it that there might be a maximum or minimum here, which of course there is.

We can see that $f''(x) = -\sin(x) < 0$ at this point indicating by the 2nd derivative test that this is a maximum.

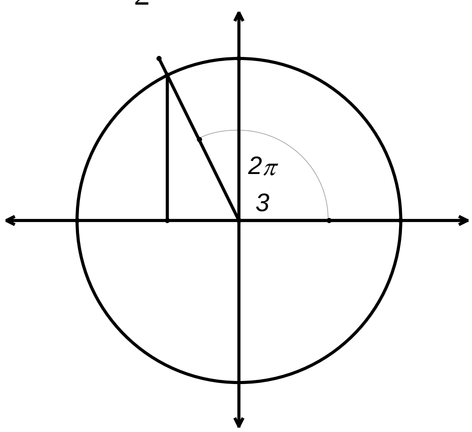
Similarly, we see at $\frac{3\pi}{2}$ that the cos(x)=0 indicating a critical point, and that -sin(x) is positive so by the 2nd derivative test, the function has a minimum.

At $x = 0$ and $x = \pi$ $f''(x) = 0$ where we can see the concavity of $\sin(x)$ change direction.

Example:

Find the local maximum and minimum values of the function
$g(x) = x + 2sin(x)$ on the interval $0 \le x \le 2\pi$

$g'(x) = 1 + 2cos(x)$ so, we have critical points at $1+2cos(x) = 0$ or
$\cos(x) = \frac{-1}{2}$



Solving this we find $x = \frac{2\pi}{3}, \frac{4\pi}{3}$

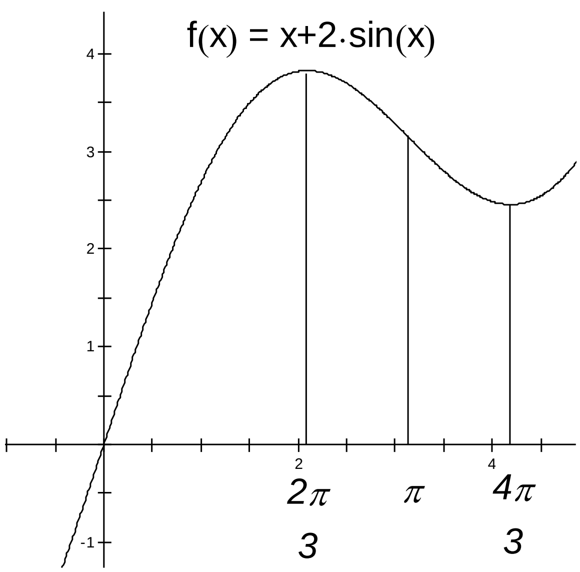
The second derivative is $-2\sin(x)$ and we find that $-2\sin\left(\frac{2\pi}{3}\right) = -2\sqrt{3}$

So, by the second derivative test, this is a maximum.

We also find that $-2\sin\left(\frac{4\pi}{3}\right) = 2\sqrt{3}$ so, by the second derivative test, this is a minimum.

Finally, since $f''(x) = -2\sin(x)$ is zero at π we see that there is a point of inflection there.

We can confirm this looking at the graph.



Example:

Find the local maximum and minimums, regions of concavity and points of inflection of

$f(x) = x^4 - 4x^3$

First, we find the derivatives

$f'(x) = 4x^3 - 12x^2$

$f''(x) = 12x^2 - 24x$

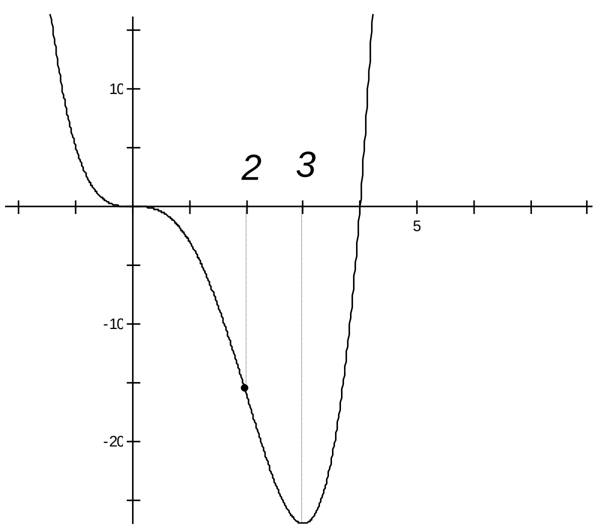
The critical points are at $4x^3 - 12x^2 = 4x^2(x-3) = 0$ so, they are 0, and 3.

At these points $f''(0) = 0$ $f''(3) = 36$ so at 0 we may have a point of inflection and at 3 we expect to have a minimum.

We can also check where

$$f''(x) = 12x^2 - 24x = 0$$
$$x(x-2) = 0$$

So, there is an additional point of inflection at $x = 2$.



Exercises

Describe the concavity of the graph and find the points of inflection (if any).

1.* $\frac{1}{x}\cdot$

2. $x + \frac{1}{x}\cdot$

3.* $x^3 - 3x + 2.$

4. $2x^2 - 5x + 2.$

5.* $\frac{1}{4}x^4 - \frac{1}{2}x^2.$

6. $x^3(1 - x).$

7.* $\frac{x}{x^2 - 1}\cdot$

8. $\frac{x + 2}{x - 2}\cdot$

9.* $(1 - x)^2(1 + x)^2.$

10. $\frac{6x}{x^2 + 1}\cdot$

11.* $\frac{1 - \sqrt{x}}{1 + \sqrt{x}}\cdot$

12. $(x - 3)^{1/5}.$

13.* Find d given that $(d, f(d))$ is a point of inflection of

$$f(x) = (x - a)(x - b)(x - c).$$

For each of the following functions (i) find the critical points, (ii) find and cla extreme values (local, end-point, absolute), (iii) indicate where the function is i and where it is decreasing, (iv) indicate the concavity of the graph, (v) specify tl of inflection, and then (vi) sketch the graph.

1.* $f(x) = (x - 2)^2.$

2. $f(x) = 1 - (x - 2)^2.$

3.* $f(x) = x^3 - 2x^2 + x + 1.$

4. $f(x) = \frac{2}{3}x^3 - \frac{1}{2}x^2 - 10x - 1.$

5.* $f(x) = \frac{x}{3x - 1}\cdot$

6. $f(x) = \frac{2x}{4x - 3}\cdot$

7.* $f(x) = \frac{x^2}{3x + 1}\cdot$

8. $f(x) = \frac{2x^2}{-x + 1}\cdot$

9.* $f(x) = \frac{x}{(3x + 1)^2}\cdot$

10. $f(x) = \frac{2x}{(x + 1)^2}\cdot$

11.* $f(x) = 3x^5 + 5x^3.$

12. $f(x) = 3x^4 + 4x^3.$

13.* $f(x) = 1 + (x - 2)^{4/3}.$

14. $f(x) = 1 + (x - 2)^{5/3}.$

15.* $f(x) = x^2(1 + x)^2.$

16. $f(x) = x^2(1 + x)^3.$

17.* $f(x) = x\sqrt{1 - x}.$

18. $f(x) = \sqrt{x - x^2}.$

Example:

Let's find the local maximums in $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$