Math 109 Calc 1 Lecture 21

Derivatives and the Shape of Graphs

Section 4.3

Recall how we can tell if a function is increasing or decreasing.

If $f(x) > 0$ on an interval, the function is increasing on the interval.
If $f(x) < 0$ on an interval, the function is decreasing on the interval.

The book calls this the increasing/decreasing test.

Example:

Find where $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing and decreasing.

We first find the derivative

$f'(x) = 12x^3 - 12x^2 - 24x = 12x(x-1)(x+1)$

We can see immediately that the derivative has 3 zeros at 0,1, and -1, where there will be local maximums and minimums.

So, we just need to look at the intervals $(-\infty,-1),(-1,0),(0,1)\ and\ (1,\infty)$

For $(-\infty,-1)$ we can try -2.
$f'(x) = -12 \cdot 8 - 12 \cdot 4 + 24 \cdot 2 = -96$
So, on this interval the function is decreasing.

You could skip this step by noticing that the coefficient of x^4 is positive and realizing that since the function is even, as *x* goes negative, the function will get very large. So, as it gets smaller from the left it will decrease.

For $(-1,0)$ we can try -1/2.
$f'(x) = -12/8 - 12/4 + 24/2 = 7.5$
So, on this interval the function is increasing.

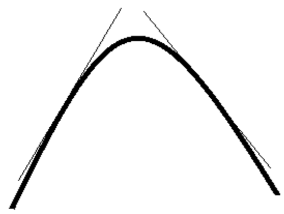
For $(0,1)$ we can try 1/2.
$f'(x) = -12/8 - 12/4 - 24/2 = -16.5$
So, on this interval the function is decreasing.

For $(0,\infty)$ we can try 2.
$f'(x) = -12 \cdot 8 - 12 \cdot 4 - 24 \cdot 2 = -96$
So, on this interval the function is increasing.
Again, you could skip this step knowing that the coefficient of the x^4 is positive.

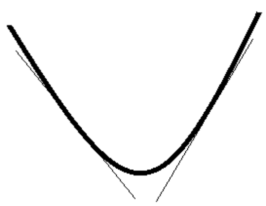
The first derivative test can tell us if a critical point is a maximum or a minimum.

The first Derivative Test

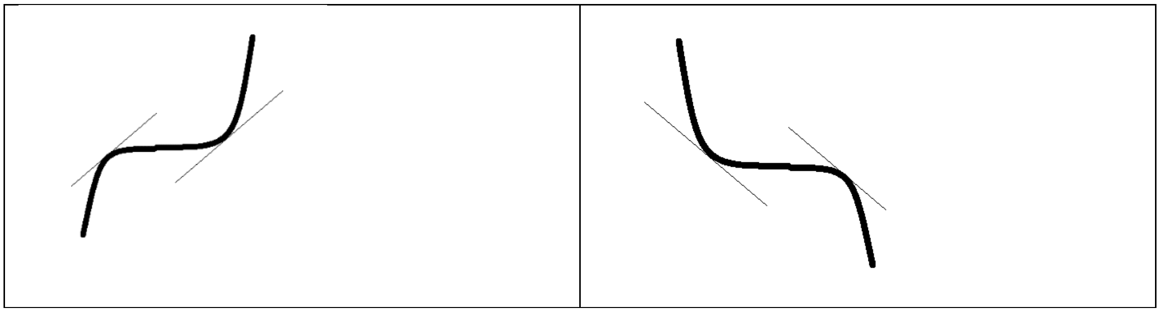
If *c* is a critical point and the derivative changes from positive to negative, then the function has a maximum at *c*.



If *c* is a critical point and the derivative changes from negative to positive, then the function has a minimum at *c*.



If *c* is a critical point and the derivative changes from negative to negative or positive to positive, then the function has no maximum or minimum.



Example:
Let's find the local maximums in $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$

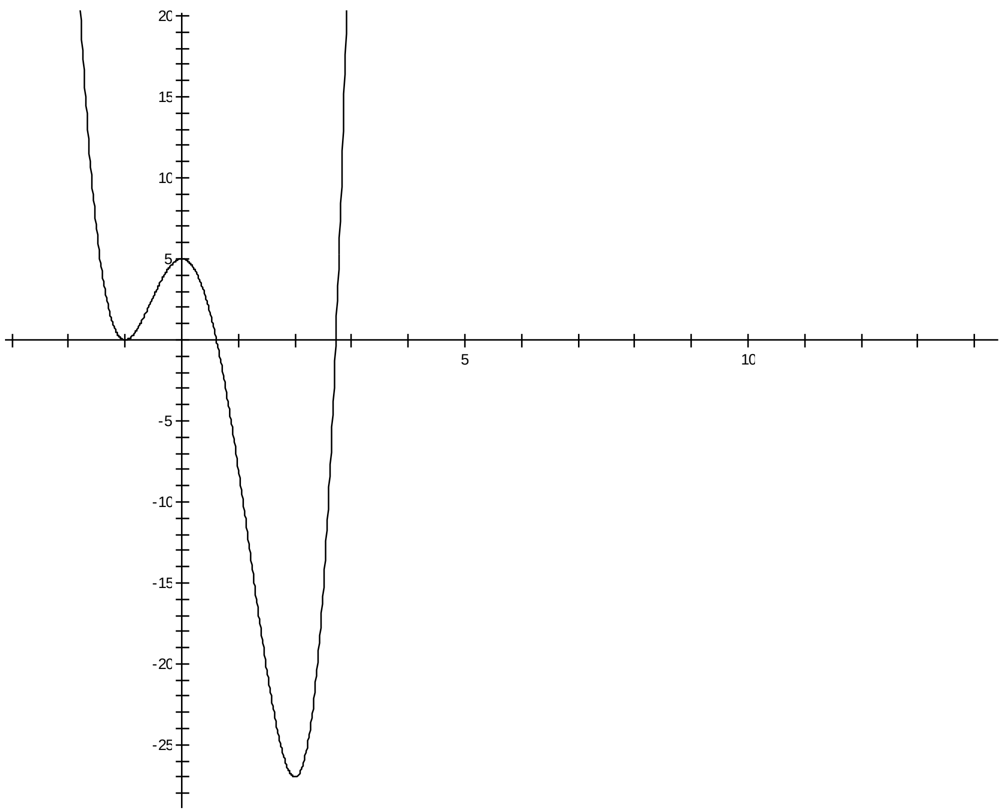
The critical points are -1, 0, and 1.

The function changes from -2 to -1/2 from negative to positive, so at -1 we have a minimum.

The function changes from -1/2 to +1/2 from positive to negative so at 0 we have a maximum.

The function changes from +1/2 to 2 from negative to positive, so at 1 we have a minimum.

The graph confirms this:



Example:

Find the local maximum and minimum values of the function
$g(x) = x + 2sin(x)$ on the interval $0 \leq x \leq 2\pi$

$g'(x) = 1 + 2cos(x)$ so, we have critical points at $1{+}2cos(x) = 0$ or

$$cos(x) = -\frac{1}{2}$$

Solving this we find $x = \frac{2\pi}{3}, \frac{4\pi}{3}$

$$0 \leq x \leq \frac{2\pi}{3}$$

We can try $\frac{\pi}{3}$ and we find that $g'\left(\frac{\pi}{3}\right) = 1 + 2cos\left(\frac{\pi}{3}\right) = 2 > 0$
So, on this interval the function is increasing.

$$\frac{2\pi}{3} \leq x \leq \frac{4\pi}{3}$$

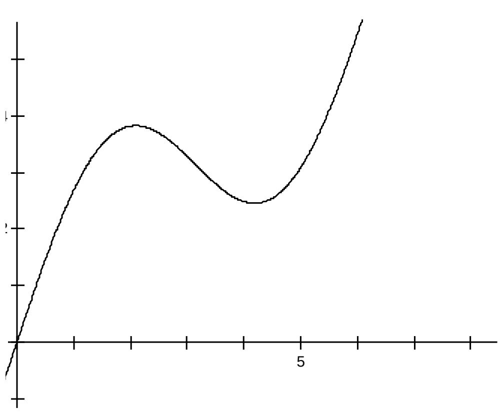
We can try π and we find that $g'(\pi) = 1 + 2cos(\pi) = -1 < 0$
So, on this interval the function is decreasing.

$$\frac{4\pi}{3} \leq x \leq 2\pi$$

We can try π and we find that $g'\left(\frac{5\pi}{3}\right) = 1 + 2cos\left(\frac{5\pi}{3}\right) = 2 > 0$
So, on this interval the function is increasing.

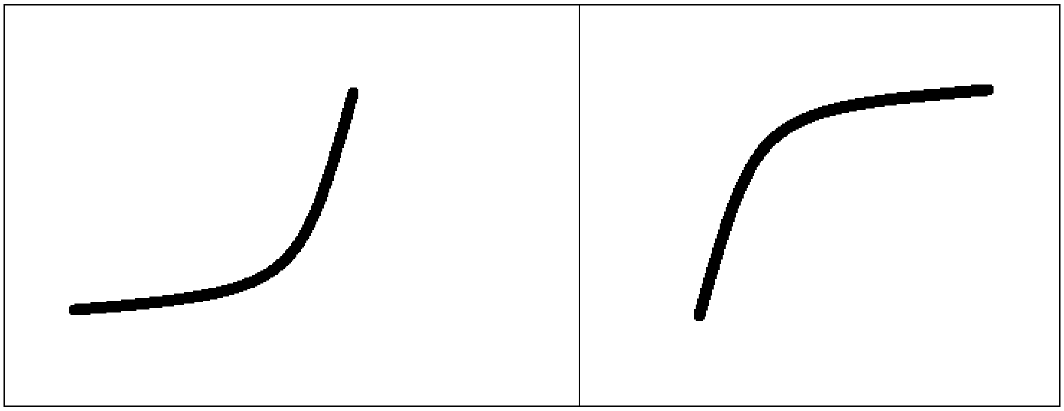
So, we conclude from the first derivative test that the function has a maximum at $\frac{2\pi}{3}$ and minimum at $\frac{4\pi}{3}$.

Again we can confirm this looking at the graph.



What the Second Derivative Tells Us About a Function

Taking a look at two graphs, consider the difference



These are both increasing functions; however, the slopes are increasing.

That is to say $f''(x) > 0$

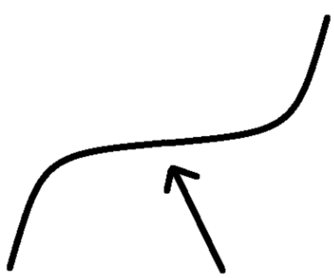
This is called concave up.

For the curve on the right the slopes are decreasing.

That is to say $f''(x) < 0$

This is called concave down.

For a curve like this, what happens where the arrow is pointing to?



$f''(x) = 0$

When this happens and the first derivative goes from positive to negative or negative to positive, this is called a **point of inflection**.

The Second Derivative Test

If a function *f* has a second derivative that is continuous near *c*,

If $f'(c) = 0$ *and* $f''(c) > 0$ then *f* has a local minimum at *c*.

If $f'(c) = 0$ *and* $f''(c) < 0$ then *f* has a local maximum at *c*.

Example:

Find the local maximum and minimums, regions of concavity and points of inflection of

$f(x) = x^4 - 4x^3$

First, we find the derivatives

$f'(x) = 4x^3 - 12x^2$

$f''(x) = 12x^2 - 24x$

The critical points are at $4x^3 - 12x^2 = 4x^2(x-3) = 0$ so, they are 0, and 3.

$f''(0) = 0$ and $f''(3) = 36$ and

That tells us that 0 is not a maximum or minimum but 3 is a minimum.

For $x < 0$ $f''(-1) = 12 + 24 > 0$ so the function is concave up.

For $0 < x < 3$ $f''(11) = 12 - 24 < 0$ so the function is concave down.

For $3 < x$ $f''(4) = 96 > 0$ so the function is concave up.

Since the first derivative changes sign at both 0 and 3, they are both points of concavity.

Exercises

Describe the concavity of the graph and find the points of inflection (if any).

1.* $\frac{1}{x}\cdot$

2. $x + \frac{1}{x}\cdot$

3.* $x^3 - 3x + 2.$

4. $2x^2 - 5x + 2.$

5.* $\frac{1}{4}x^4 - \frac{1}{2}x^2.$

6. $x^3(1 - x).$

7.* $\frac{x}{x^2 - 1}\cdot$

8. $\frac{x + 2}{x - 2}\cdot$

9.* $(1 - x)^2(1 + x)^2.$

10. $\frac{6x}{x^2 + 1}\cdot$

11.* $\frac{1 - \sqrt{x}}{1 + \sqrt{x}}\cdot$

12. $(x - 3)^{1/5}.$

13.* Find d given that $(d, f(d))$ is a point of inflection of

$$f(x) = (x - a)(x - b)(x - c).$$

For each of the following functions (i) find the critical points, (ii) find and cla extreme values (local, end-point, absolute), (iii) indicate where the function is i and where it is decreasing, (iv) indicate the concavity of the graph, (v) specify tl of inflection, and then (vi) sketch the graph.

1.* $f(x) = (x - 2)^2.$

2. $f(x) = 1 - (x - 2)^2.$

3.* $f(x) = x^3 - 2x^2 + x + 1.$

4. $f(x) = \frac{2}{3}x^3 - \frac{1}{2}x^2 - 10x - 1.$

5.* $f(x) = \frac{x}{3x - 1}\cdot$

6. $f(x) = \frac{2x}{4x - 3}\cdot$

7.* $f(x) = \frac{x^2}{3x + 1}\cdot$

8. $f(x) = \frac{2x^2}{-x + 1}\cdot$

9.* $f(x) = \frac{x}{(3x + 1)^2}\cdot$

10. $f(x) = \frac{2x}{(x + 1)^2}\cdot$

11.* $f(x) = 3x^5 + 5x^3.$

12. $f(x) = 3x^4 + 4x^3.$

13.* $f(x) = 1 + (x - 2)^{4/3}.$

14. $f(x) = 1 + (x - 2)^{5/3}.$

15.* $f(x) = x^2(1 + x)^2.$

16. $f(x) = x^2(1 + x)^3.$

17.* $f(x) = x\sqrt{1 - x}.$

18. $f(x) = \sqrt{x - x^2}.$