Math 109 Calc 1 Lecture 21

Derivatives and the Shape of Graphs

Section 4.3

The tools we have to connect an equation to a graph.

The roots are where $f(x)=0$

If $f'(x)>0$ on an interval, then the function is increasing,

If $f'(x)<0$ on an interval, then the function is decreasing

If $f'(x)=0$ at a point then function may have a maximum, minimum or neither at this point.

The first derivative test

If $f'(x)$ goes from + to – at c, then there is a maximum

If $f'(x)$ goes from – to + at c, then there is a minimum

Concavity

If $f''(x)>0$ then $f'(x)$ is increasing so the graph is concave up

If $f''(x)<0$ then $f'(x)$ is decreasing so the graph is concave down

The Second Derivative Test

If a function $f(x)$ has a second derivative that is continuous near c,

If $f'(c) = 0 \; and \; f''(c) > 0$ then $f(x)$ has a local minimum at c.

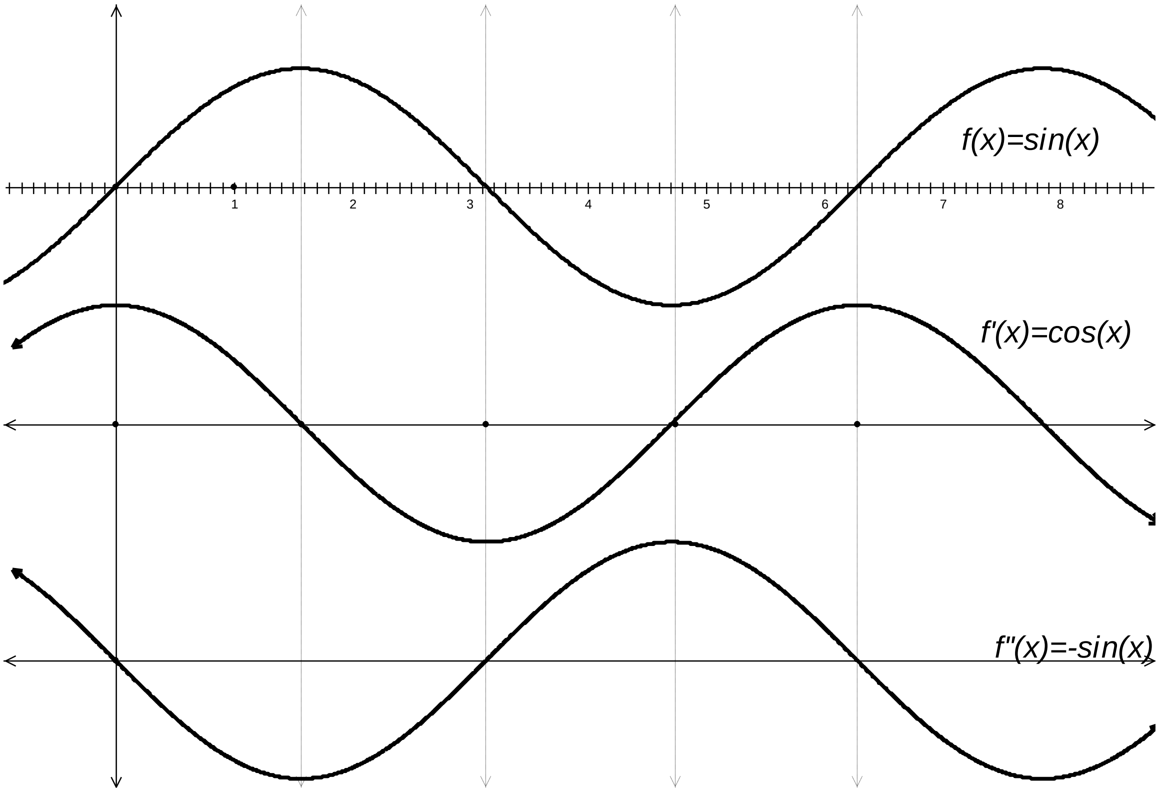
If $f'(c) = 0 \; and \; f''(c) < 0$ then $f(x)$ has a local maximum at c.

If $f'(c) = 0 \; and \; f''(c) = 0$ then $f(x)$ has a point of inflection at c.

If $f''(x)=0$ at a point, then it is a point of inflection where the function switches concavity

Example:

$f(x) = \sin(x)$
$f'(x) = \cos(x)$
$f''(x) = -\sin(x)$



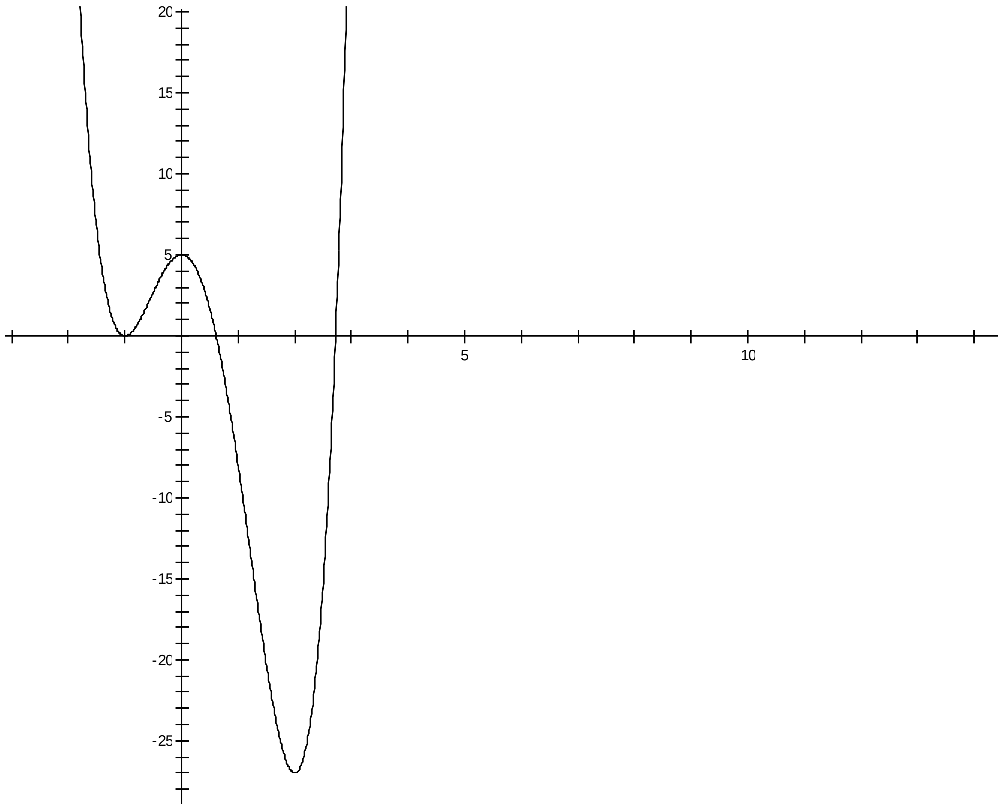
Note the roots of $f(x)$ at 0, π, 2π

At $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ $f'(x) = 0$ so the function can have a maximum and minimum.
On the interval $(0, \pi)$ $f''(x)$ is negative so the function is concave down
On the interval $(\pi, 2\pi)$ $f''(x)$ is positive so the function is concave up
At $\frac{\pi}{2}$ $f''(x)$ is negative so the function has a maximum
At $\frac{3\pi}{2}$ $f''(x)$ is positive so the function has a minimum
At 0, π, 2π $f''(x) = 0$ so the function has a point of inflection.

Example:

Let's look at the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$

The graph looks like this.



Using the rational root theorem, we can find that -1 is a root twice.

$$f(x) = 3x^4 - 4x^3 - 12x^2 + 5 = (x+1)^2 (3x^2 - 10x + 5)$$

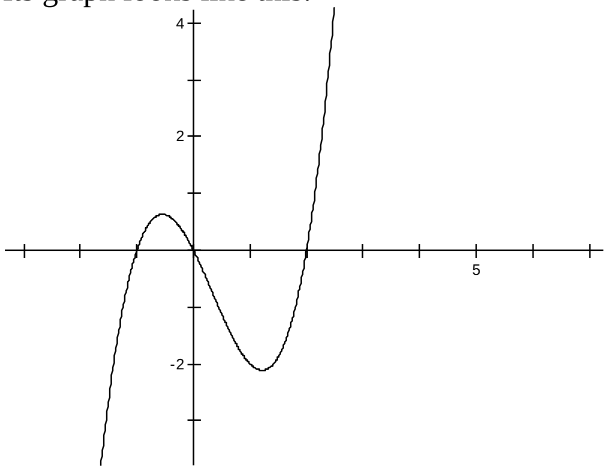
Using the quadratic formula, we can find the remaining two roots

$$\frac{5 \pm \sqrt{10}}{3} \approx .613 \text{ and } 2.72$$

The derivative is

$$f'(x) = 12x^3 - 12x^2 - 24x = 12x(x-2)(x+1)$$

Its graph looks like this:



On the interval $(-\infty, -1)$ $f'(x) < 0$ so the function is decreasing

On the interval $(-1, 0)$ $f'(x) > 0$ so the function is increasing

On the interval $(0, 2)$ $f'(x) < 0$ so the function is decreasing

On the interval $(2, \infty)$ $f'(x) > 0$ so the function is increasing

The derivative has roots at {-1,0,2} so these are critical points.

The derivative goes from – to + at -1 so this is a minimum

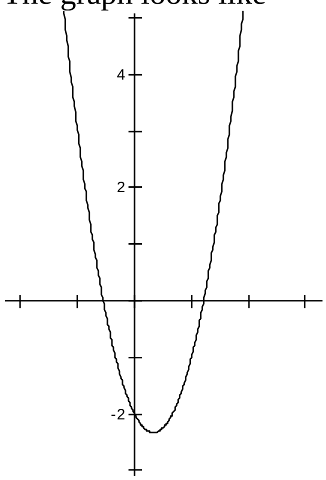
The derivative goes from + to - at 0 so this is a maximum

The derivative goes from – to + at 2 so this is a minimum

The second derivative is

$$f''(x) = 36x^2 - 24x - 24 = 12(3x^2 - 2x - 2)$$

The graph looks like



The roots are at

$$\frac{1 \pm \sqrt{7}}{3} \approx -.55,\ 1.21$$

So, these are points of inflection on $f(x)$

On the interval $(-\infty, -.55)$

If $f''(x) < 0$ so the graph is concave down

On the interval $(-.55, 1.21)$

If $f''(x) > 0$ so the graph is concave up

On the interval $(1.21, \infty)$

If $f''(x) > 0$ so the graph is concave up

Also note that

$f''(-1) = 12 > 0$ so, this is a minimum

$f''(0) = -24 < 0$ so, this is a maximum

$f''(2) = 6 > 0$ so, this is a minimum

Exercises

Describe the concavity of the graph and find the points of inflection (if any).

1.* $\frac{1}{x}$.

2. $x + \frac{1}{x}$.

3.* $x^3 - 3x + 2$.

4. $2x^2 - 5x + 2$.

5.* $\frac{1}{4}x^4 - \frac{1}{2}x^2$.

6. $x^3(1 - x)$.

7.* $\frac{x}{x^2 - 1}$.

8. $\frac{x + 2}{x - 2}$.

9.* $(1 - x)^2(1 + x)^2$.

10. $\frac{6x}{x^2 + 1}$.

11.* $\frac{1 - \sqrt{x}}{1 + \sqrt{x}}$.

12. $(x - 3)^{1/5}$.

13.* Find d given that $(d, f(d))$ is a point of inflection of

$$f(x) = (x - a)(x - b)(x - c).$$

For each of the following functions (i) find the critical points, (ii) find and cla extreme values (local, end-point, absolute), (iii) indicate where the function is ii and where it is decreasing, (iv) indicate the concavity of the graph, (v) specify tl of inflection, and then (vi) sketch the graph.

1.* $f(x) = (x - 2)^2$.

2. $f(x) = 1 - (x - 2)^2$.

3.* $f(x) = x^3 - 2x^2 + x + 1$.

4. $f(x) = \frac{2}{3}x^3 - \frac{1}{2}x^2 - 10x - 1$.

5.* $f(x) = \frac{x}{3x - 1}$.

6. $f(x) = \frac{2x}{4x - 3}$.

7.* $f(x) = \frac{x^2}{3x + 1}$.

8. $f(x) = \frac{2x^2}{-x + 1}$.

9.* $f(x) = \frac{x}{(3x + 1)^2}$.

10. $f(x) = \frac{2x}{(x + 1)^2}$.

11.* $f(x) = 3x^5 + 5x^3$.

12. $f(x) = 3x^4 + 4x^3$.

13.* $f(x) = 1 + (x - 2)^{4/3}$.

14. $f(x) = 1 + (x - 2)^{5/3}$.

15.* $f(x) = x^2(1 + x)^2$.

16. $f(x) = x^2(1 + x)^3$.

17.* $f(x) = x\sqrt{1 - x}$.

18. $f(x) = \sqrt{x - x^2}$.