Math 109 Calc 1 Lecture 2

Limits

The Idea of a Limit

We start with a function *f* defined near a number *c* but not necessarily at *c*.
We have a number *l* which is intended to make you think of the word Limit.

We write the following:

→approaches

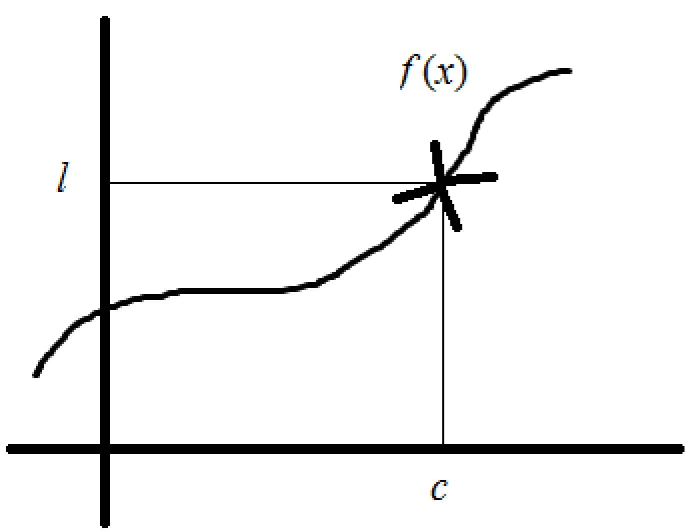
$\lim_{x \to c} f(x) = l$ or alternatively $f(x) \to l$ as $x \to c$

We can read this as, the limit of *f* (*x*) as *x* approaches *c* is *l*.

We could also say, as *x* approaches *c*, *f* (*x*) approaches *l*.

Or

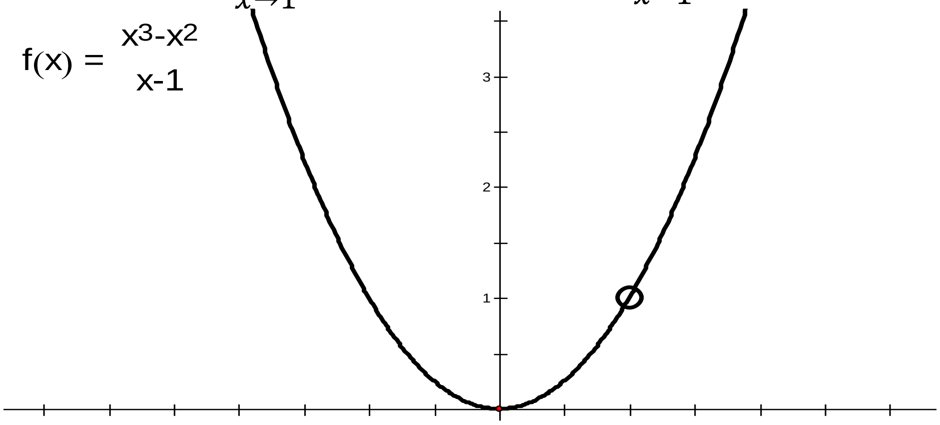
If *x* is approximately equal to *c* then *f* (*x*) is approximately equal to *l*.



Note in taking the limit as *x* approaches *c* that it does not matter if *f* is defined at *c* nor what it is defined as there. It is however required that close to *c* the function must be defined everywhere.

What matters is how *f* is defined near *c*.

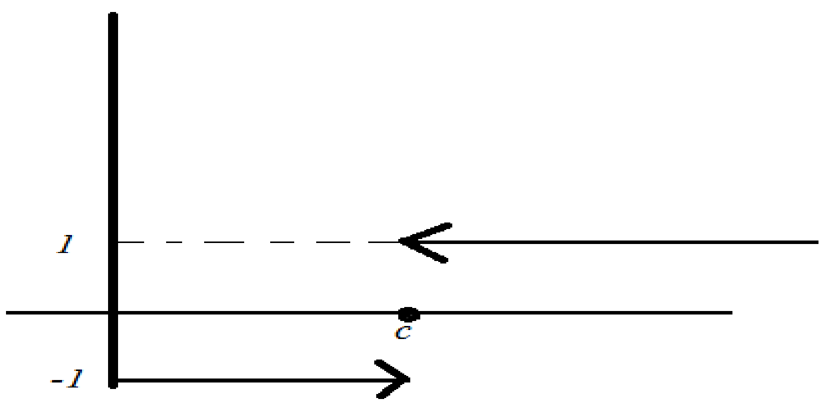
Take for example $\lim_{x \to 1} f(x)$ where $f(x) = \frac{x^3 - x}{x-1}$



Note that the function is not defined at 1, however the limit still exists, because the closer *x* gets to 1, the closer *f* (*x*) gets to 1.

Now consider a very different situation

$$f(x) = \begin{cases} 1, x > c \\ -1, x < c \end{cases}$$



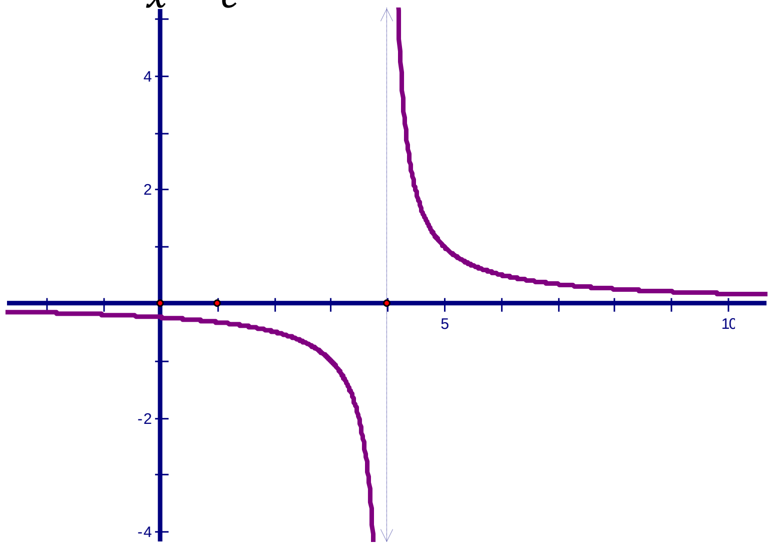
As *x* approaches *c* from the left, *f* (*x*) approaches -1, but as *x* approaches *c* from the right, *f* (*x*) approaches 1.

There is no single number that *f* (*x*) approaches as *x* approaches *c* so the limit of

$\lim_{x \to c} f(x) = l$ does not exist

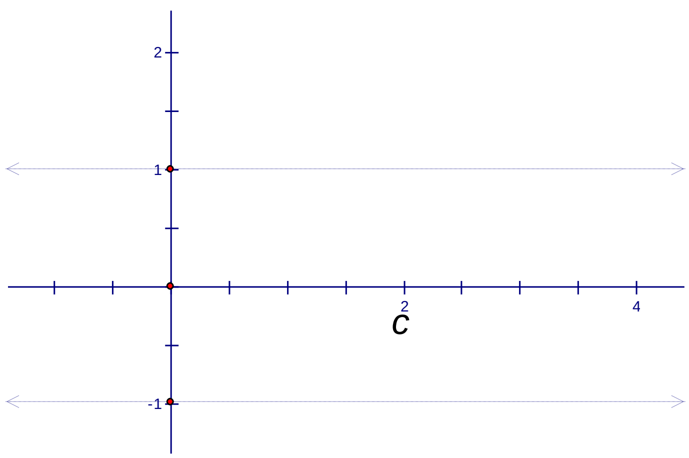
There are other circumstances in which a limit does not exist.

$$f(x) = \frac{1}{x - c}$$



Here as *x* approaches *c* from the right, *f* (*x*) gets arbitrarily large, but as it approaches *c* from the left it gets arbitrarily large negatively. Clearly there is no *l* that *f* (*x*) approaches as *x* approaches *c*.
A last and most interesting example of a function with no limit is

$$f(x) = \begin{cases} 1, x\ rational \\ -1, x\ irrational \end{cases}$$



In this example as *f* (*x*) approaches *c* the value of *f* (*x*) jumps wildly between 1 and -1, so this function does not have a limit at *c*.
Note that here the value of *f* (*c*) is clearly defined to be either 1 or -1, but yet there is no limit.
Clearly a limit has to do with the behavior of the function near *c* but not at *c*.
To go further we need a precise definition of a limit.

Recap:

We are looking at the limit of a function $f(x)$ at a point c.

We are interested in the behavior of the function close to c.

The value of the function at c if it exists at all is unimportant when looking at the limit.

Section 2.2 The Definition of a Limit

I will show you this definition, how it works, and it will be used in establishing some rules for finding limits, however

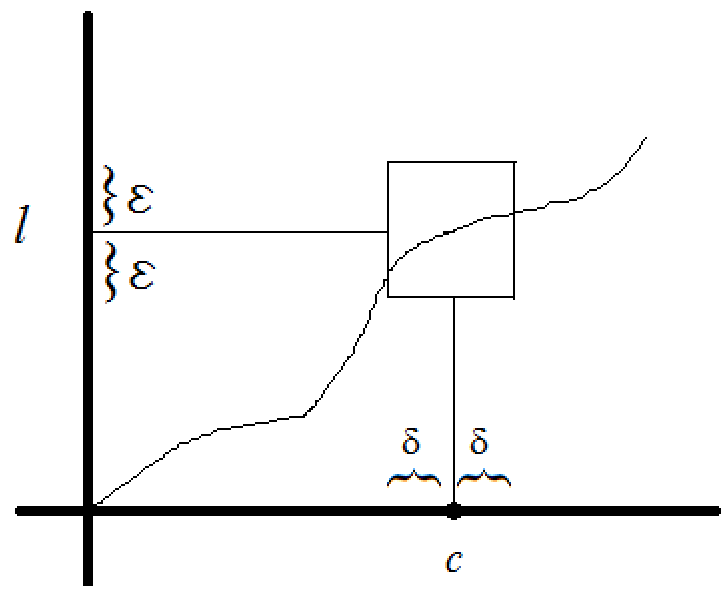
YOU WILL NOT BE RESPONSIBLE FOR THIS DEFINITION ON AN EXAM.

For those of you who go on in mathematics, you will come across this definition in a course called Analysis.

The Definition of a Limit

$$\lim_{x\to c} f(x) = l \ if \begin{cases} for\ each\ \epsilon > 0\ there\ exists\ a\ \delta > 0\ such\ that \\ if\ 0 < |x-c| < \delta\ then\ |f(x)-l| < \epsilon \end{cases}$$

To illustrate this definition, we begin with a graph and two values *c* and *l*.

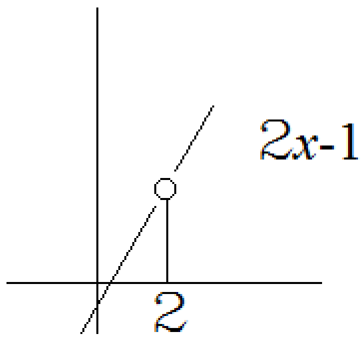


That is, given an ϵ, find an interval around c of width δ such that our function must be within ϵ of our limit *l*.

An Example:

Suppose we wish to show that

$$\lim_{x \to 2}(2x - 1) = 3$$



Because of the way the definition is worded, it is best to work backwards

We need to find a $\delta > 0$ such that if $0 < |x\text{-}2| < \delta$ that $|f(x) - 3| < \epsilon$, or more specifically that

$|(2x\text{-}1)\text{-}3| < \epsilon$

We rewrite $|(2x\text{-}1)\text{-}3| = |2x\text{-}4| = 2|x\text{-}2|$

So, if we choose $\delta = \frac{\varepsilon}{2}$ then we have working in reverse

$$0 < |x - 2| < \frac{\varepsilon}{2}$$

$$0 < 2|x - 2| < \varepsilon$$

$$0 < |2x - 4| < \varepsilon$$

$$0 < |(2x - 1) - 3| < \varepsilon$$

Most such proofs are more complicated and require a bit more ingenuity, however we will not see many in this class.

Note that this definition does not tell you anything about what the limit should be. It merely gives you a way to prove that a particular *l* is the limit.

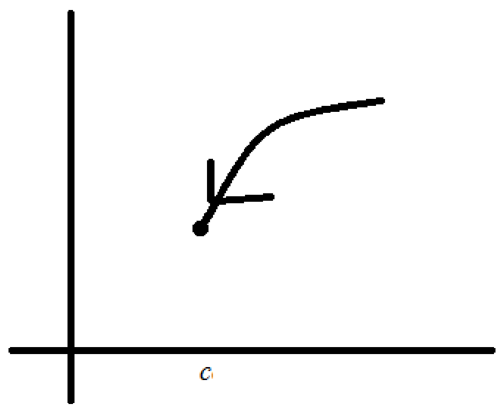
Later on, we will look at ways to find limits.

One Sided Limits

I some cases we will be interested in the limit of a function as it approaches *c* from just one direction.

We can write this as

$$\lim_{x \to c^+} f(x) = l$$

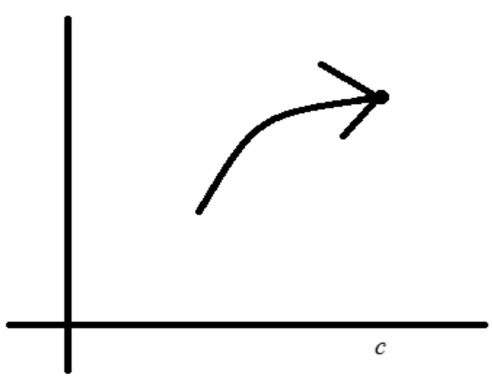


Example: $\lim_{x \to 0^+} \sqrt{x}$

To mean we approach *c* from the positive direction only or

To mean we approach *c* from the positive direction only or

$$\lim_{x \to c^-} f(x) = l$$



to mean we approach *c* from the negative direction.

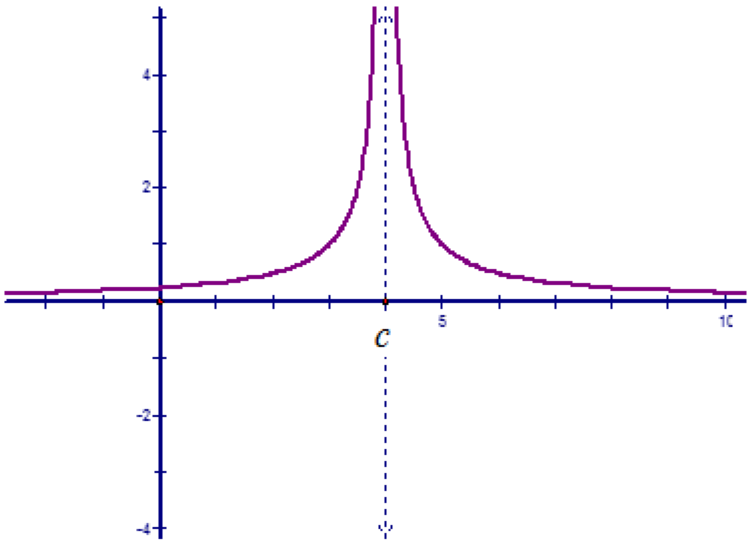
These are known as one sided limits.
A useful theorem that we shall not prove in this class is that

$$\lim_{x \to c} f(x) = l \ \ \textit{iff} \ \lim_{x \to c^+} f(x) = l \ \text{ and } \lim_{x \to c^-} f(x) = l$$

That is, proving both one sided limits is sufficient to proving the limit and vice versa.

Limits at Infinity

$$f(x) = \frac{1}{|x-c|}$$



In this example, $f(x)$ gets arbitrarily large from both directions as it approaches c, however it does not stay near any fixed number l.

Technically the limit $\lim_{x \to c} \frac{1}{|x-c|}$ DNE (does not exist), however we can indicate the particular way it does not exist by writing

$$\lim_{x \to c} \frac{1}{|x-c|} = \infty$$

This can be called an **infinite limit**.

At such a limit you will find a vertical asymptote defined by $y=c$.

Similarly, a function could get arbitrarily small, which would be written

$$\lim_{x \to c} \frac{-1}{|x-c|} = -\infty$$