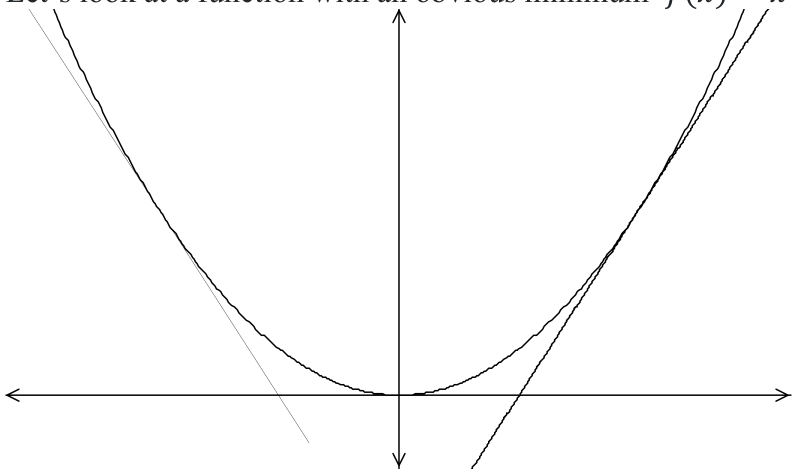
Math 109 Calc 1 Lecture 19

Maximums and Minimums

Section 4.11

One important use of the derivative is finding the maximum and minimum values of a function.

Let's look at a function with an obvious minimum $f(x) = x^2$



Note that to the left of the vertex the slope of the tangent is negative, but to the right it is positive. As a point moves from left to right its slope changes smoothly from negative to positive, achieving the value zero at the vertex.

This suggests that for a differentiable function, it has a minimum where its derivative is zero.

Example:

$f(x) = 3 - x^2$

$f'(x) = 2x$

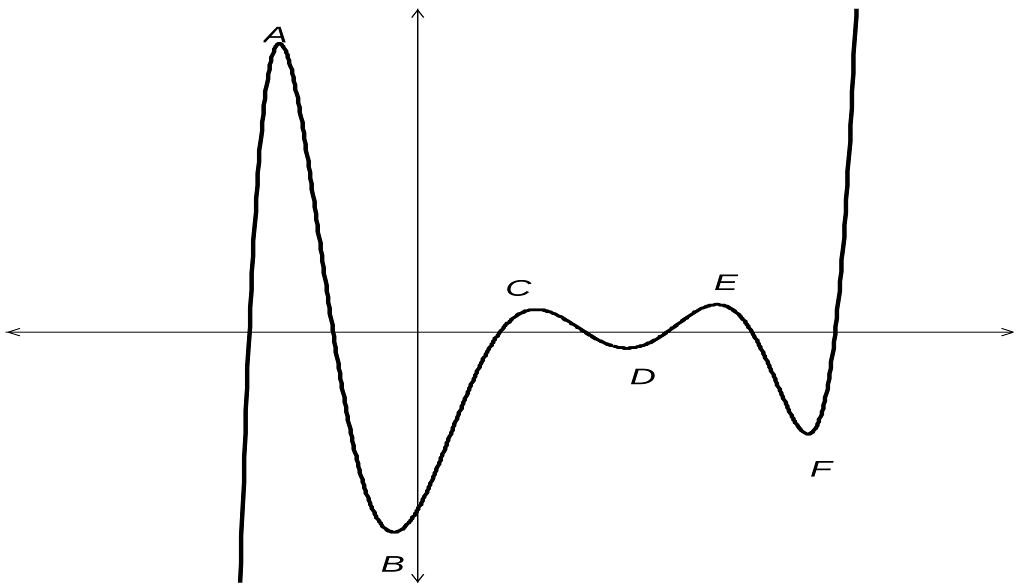
If $2x = 0$ then x = 0 then the local maximum or minimum occurs at x = 0 and is $f(0) = 3 - 0^2 = 3$.

Clearly this is a maximum.

So, it seems to be the case that maximums and minimums will occur where the derivative of a function is zero, but there are a few details that need to be examined.

Local Maximums and Minimums

First consider the function graphed below.



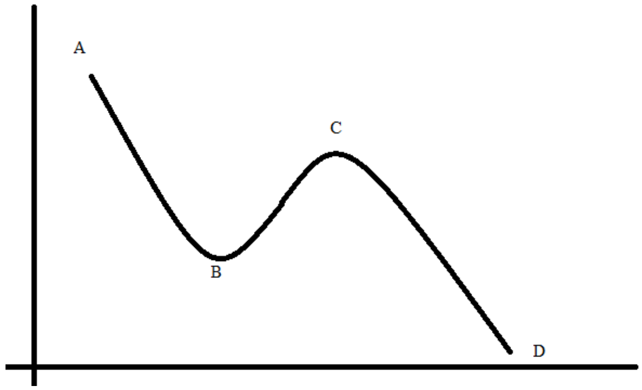
Note that it has multiple points where the derivative is zero. We can see that at those points the function has a **local** maximum or minimum.

A local maximum can be defined as $f(c)$ for which there is an interval [a,b] where for all x in $[a,b]$ $f(c) \geq f(x)$

Likewise a local minimum is $f(c)$ for which there is an interval [a,b] where for all x in [a,b] $f(c) \leq f(x)$

Absolute Maximum and Minimums

Now consider this function



It appears to have a local minimum at B and a local maximum at C, but it has a greater maximum at the endpoint A and a lessor minimum at the endpoint D.

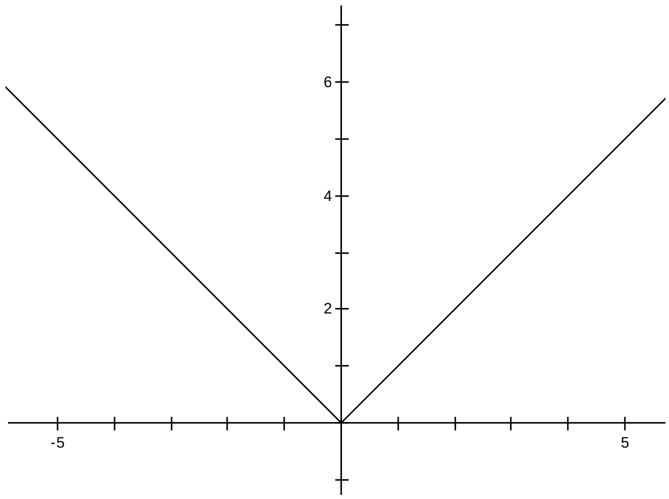
We can say that it has an **absolute** maximum at A and an **absolute** minimum at D.

An absolute maximum can be defined as $f(c)$ for where for all x in the domain of f we have $f(c) \geq f(x)$

Likewise, we have an absolute minimum $f(c)$ for where for all x in the domain of f we have $f(c) \leq f(x)$.

Critical Points

Consider the function $f(x) = |x|$



There clearly is a minimum at $f(0) = 0$. However, our local minimum test, $f'(x) = 0$ does not seem to hold.

What is important to note is that the function is not differentiable at $x = 0$. Places where the function is not differentiable or where the derivative of the function is zero are called **critical points** or **critical numbers.**

Maximums and minimums, both relative and absolute may appear at critical points.

To complete our examination of maximums and minimums there are a few important ideas to take notice of.

1. The Extreme Value Theorem:
   If a function is continuous on a closed interval $[a,b]$ then there are values c, and d on the interval where the function has an absolute maximum $f(c)$ and an absolute minimum $f(d)$.

2. Fermat's Theorem:
   If a function has a local maximum or minimum at c and if it is differentiable at c then the derivative is zero at c.

3. The Closed Interval Method for finding absolute maximum and minimums.
   To find the absolute maximum and minimum values on a continuous function on a closed interval do the following.
   a. Find the values of the function at any critical numbers in (a,b)
   b. Find the values of the function at its endpoints
   c. The largest of the values in these steps is the absolute maximum and the smallest is the minimum.

Example:

$f(x) = |x + 1| + 2$

The derivative is

$$f'(x) = \begin{cases} 1, & x > -1 \\ -1 & x < -1 \end{cases}$$

but the function is not differentiable at x = -1, so it has a critical point there.

You should note that this will usually happen where the contents of an absolute value are zero.

Example:

$f(x) = 3x^4 - 16x^3 + 18x^2$ on the interval $-1 \leq x \leq 4$

$f'(x) = 12x^3 - 48x^2 + 36x = 12x^2(x^2 - 4\text{x} + 3) = 12x^2(x - 1)(x - 3)$

So, the critical points are at x= 0, 1, and 3

$f(0) = 0$
$f(1) = 5$
$f(3) = -27$

At the end points we have
$f(-1) = 37$
$f(4) = 32$

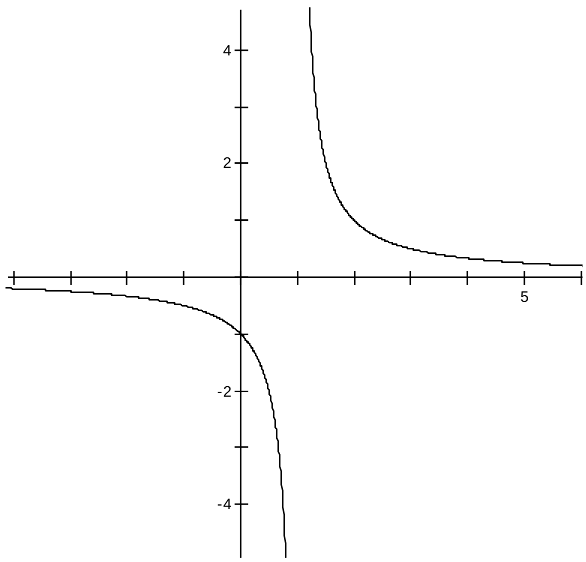
So, the absolute maximum is 37 and the absolute minimum is -27.

Example:

$$f(x) = \frac{1}{x-1}$$

$$f'(x) = \frac{-1}{(x-1)^2}$$

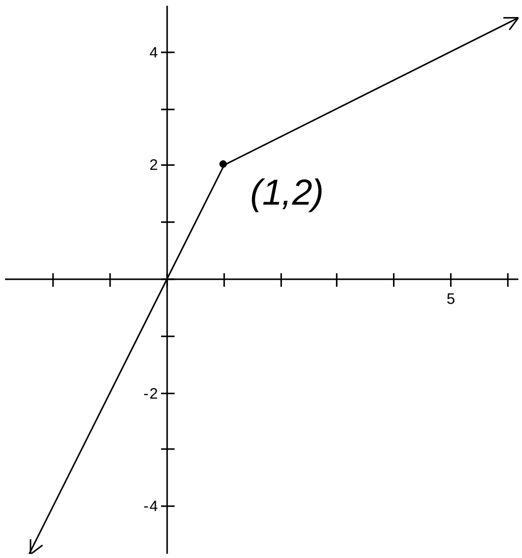
This derivative is defined everywhere on the function's domain so there are no critical points and therefore no extreme values.



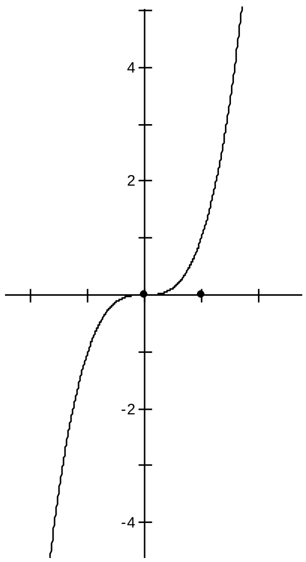
Example:

$$f(x) = \begin{cases} 2x, & x < 1 \\ \frac{1}{2}x + \frac{3}{2} & x >= 1 \end{cases}$$

This derivative has a critical point at $f(1)$, however looking at the graph you can see that there is no extreme value there.



Note: that a function defined on an open interval may not have any absolute or relative maximum or minimum, e.g. $f(x) = x^3$.



There are two things to note here.

First, the interval that this function is defined on is open, and as a consequence, there is no absolute maximum or minimum.

Furthermore, although $f'(0) = 0$, there is no relative maximum or minimum at $x = 0$.

This brings us to an important question about relative maximums and minimums. If a function is differentiable at a point c and $f'(c) = 0$, then which is it?

The Second Derivative Test

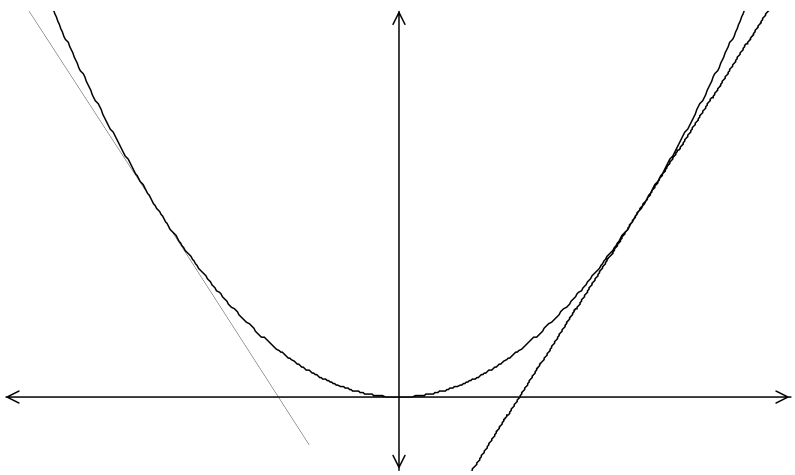
If a function is twice differentiable at a point c and $f'(c) = 0$ then the following will be true.

If $f''(c) > 0$ then the function is increasing at c and $f(c)$ is a local minimum.

If $f''(c) < 0$ then the function is decreasing at c and $f(c)$ is a local maximum.

If $f''(c) = 0$ then $f(c)$ is a neither a local minimum nor a local maximum. We call a point where $f''(c) = 0$ a **Point of Inflection**. More on this later.

Taking another look at $f(x) = x^2$



Note that as we move from left to right across the origin that the slopes of the tangents are increasing. If you forget the orientation of the second derivative test, it is a good idea to think of this example.

Examples for students to try in class:

Find the critical points and the local extreme values:

1. $x^3 + 3x - 2$
2. $\frac{2-3x}{2+x}$
3. $(1-x)^2(1+x)^2$
4. $\frac{1}{x+1} - \frac{1}{x+2}$