Math 109 Calc 1 Lecture 18

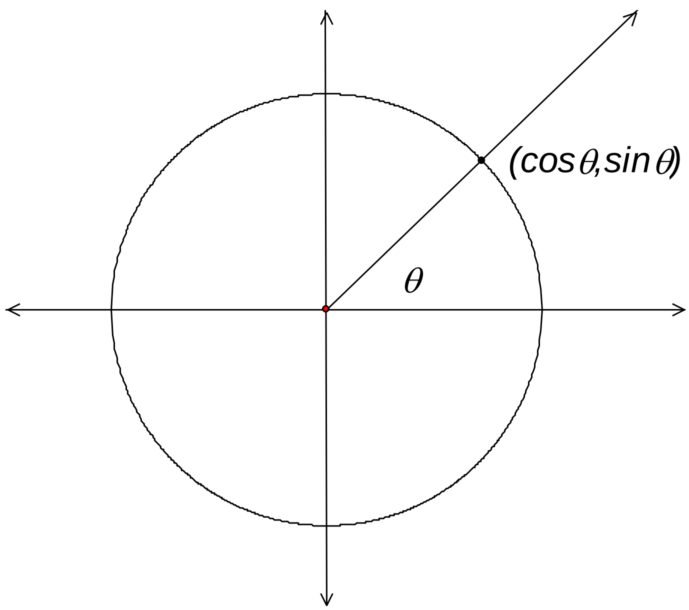
Hyperbolic Functions

Section 3.11

Circular (aka Trigonmetric Functions)

There are a number of ways to define the trigonometric functions. One way involves the use of the unit circle.

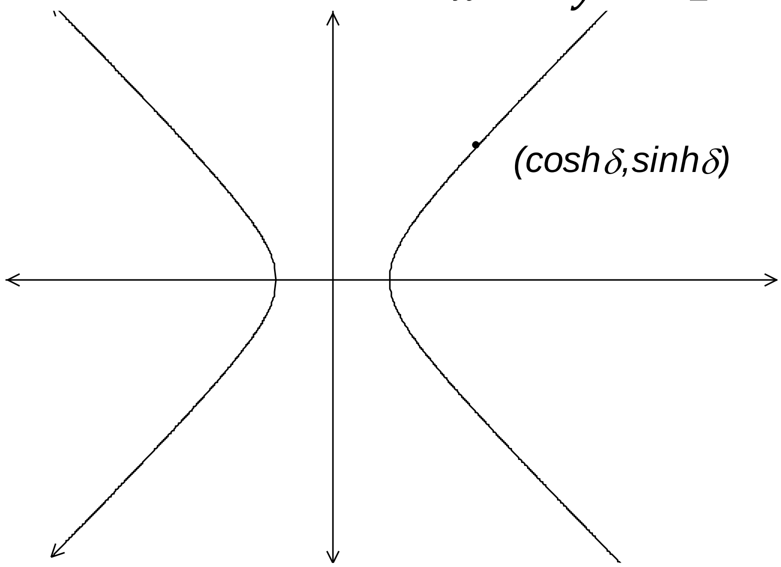
$$x^2 + y^2 = 1$$



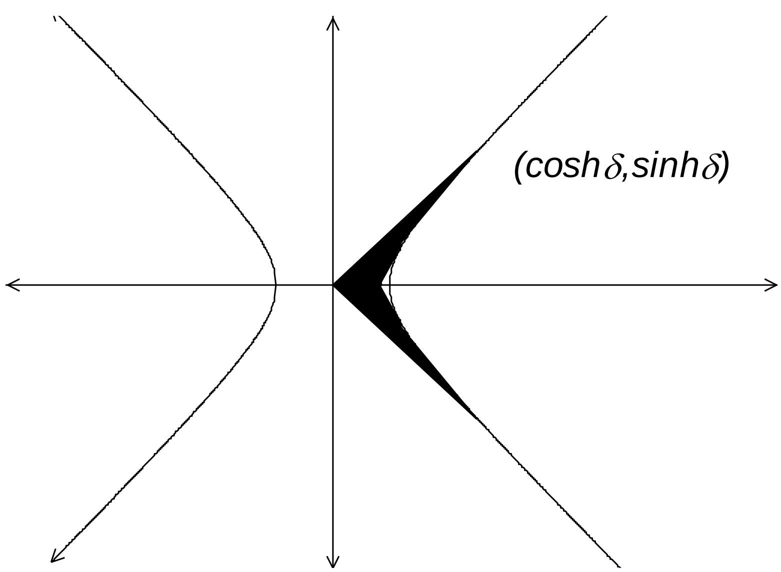
The coordinates of the point intercepted by a ray from the center with angle θ are defined as the cosine and sine of the angle.

In a similar manner we can use a unit hyperbola to define new functions.

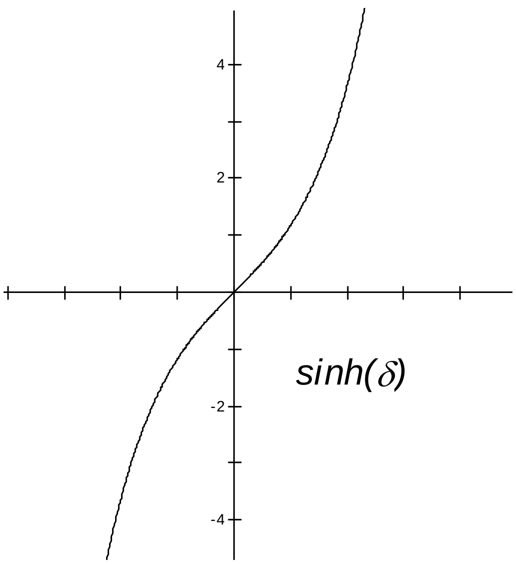
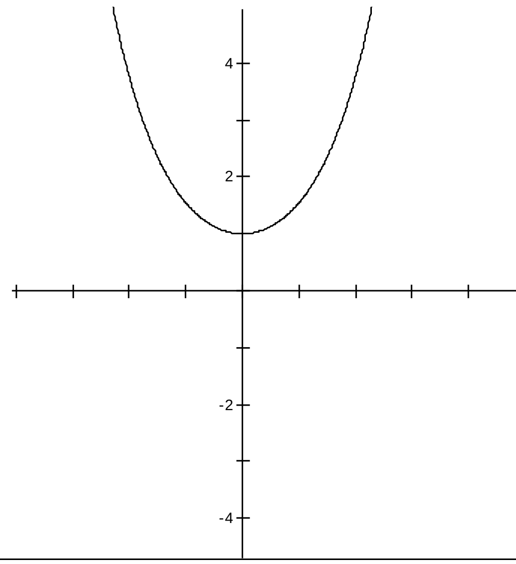
$$x^2 - y^2 = 1$$



If you set δ (delta) equal to the interior area shown below, called a hyperbolic angle

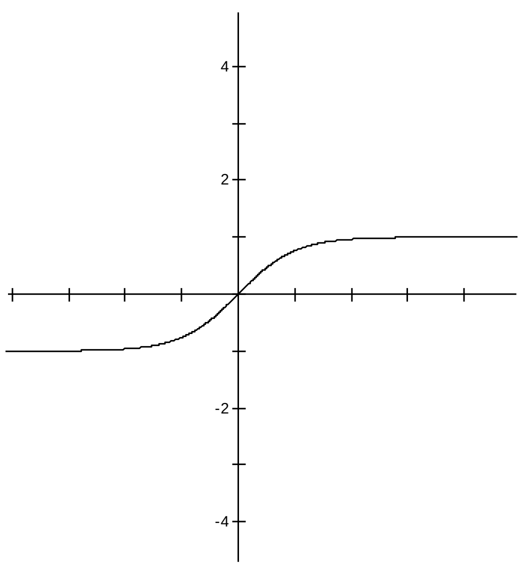


Then you can express the hyperbolic functions in terms of e^x

Hyperbolic Sine	Hyperbolic Cosine
$sinh(\delta) = \frac{e^{\delta} - e^{-\delta}}{2}$	$cosh(\delta) = \frac{e^{\delta} + e^{-\delta}}{2}$
	

We can also define the hyperbolic tangent as

$$tanh(\delta) = \frac{sinh(\delta)}{cosh(\delta)} = \frac{e^{\delta} - e^{-\delta}}{e^{\delta} + e^{-\delta}}$$



There are some remarkably similar identities and formulae associated with these functions.

$sin^2(x) + cos^2(x) = 1$	$cosh^2(x) - sinh^2(x) = 1$
$1 + tan^2(x) = sec^2(x)$	$1 - tanh^2(x) = sech^2(x)$

The sum formulae are similar

$sin(x + y) =$ $sin(x)cos(y) + cos(x)sin(y)$ $cos(x + y) =$ $cos(x)cos(y) - sin(x)sin(y)$	$sinh(x + y) =$ $sinh(x)cosh(y) + cosh(x)sinh(y)$ $cosh(x + y) =$ $cosh(x)cosh(y) + sinh(x)sinh(y)$

Also note that we have similarity in odd evenness

$sin(-x) = -sin(x)$ $cos(-x) = cos(x)$	$sinh(-x) = -sinh(x)$ $cosh(-x) = cosh(x)$

Important derivatives

$[sinh(x)]' = cosh(x)$

$[cosh(x)]' = sinh(x)$

$[tanh(x)]' = sech^2(x)$

The hyperbolic functions are not periodic like the trigonometric functions, however there is another very deep similarity. The trigonometric functions sine and cosine can be expressed as follows.

$sinh(\delta) = \frac{e^{\delta} - e^{-\delta}}{2}$ $cosh(\delta) = \frac{e^{\delta} - e^{-\delta}}{2}$	$sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}$ $cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}$

Taylor Series Expansion

If you take Calc 2 you will learn about Taylor series expansion of functions. Here is a brief introduction.

Say we want to try to expand e^x in terms of a polynomial
$e^x = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$

We can start by evaluating both sides at zero.
$e^0 = a_0 = 1$
So now know that $a_0 = 1$

We find the derivative of both sides and we find that
$e^x = a_1 + 2a_2 x + 3a_3 x^2 + +4a_3 x^3 \cdots$

Again, we evaluate at zero finding that
that $a_1 = 1$

Repeating these steps, we find that

that $a_2 = 2, a_3 = 6, a_4 = 24, \cdots a_n = n!$

So, our expansion becomes

$$e^x = \frac{1}{0!}x^0 + \frac{1}{1!}x^1 + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots$$

Using this same process, we find the following expansions for sine, cosine, and their hyperbolic equivalents.

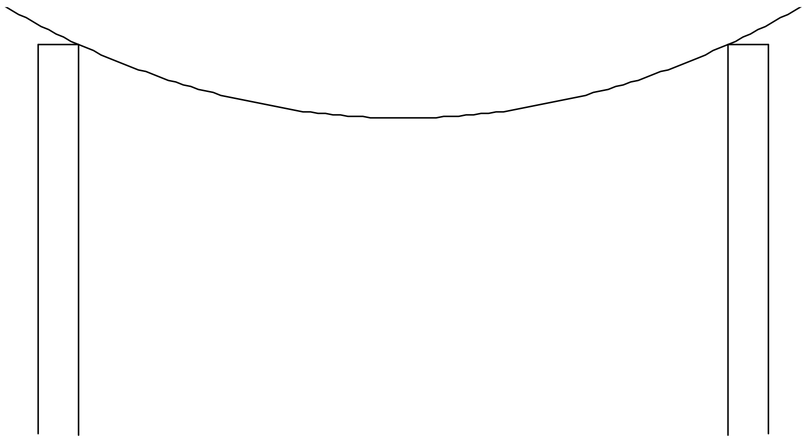
$$sin(x) = \frac{1}{1!}x^1 - \frac{3}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots$$

$$sinh(x) = \frac{1}{1!}x^1 + \frac{3}{3!}x^3 + \frac{1}{5!}x^5 + \frac{1}{7!}x^7 + \cdots$$

$$cos(x) = \frac{1}{0!}x^0 - \frac{2}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots$$

$$cosh(x) = \frac{1}{0!}x^0 + \frac{2}{2!}x^2 + \frac{1}{4!}x^4 + \frac{1}{6!}x^6 + \cdots$$

Applications

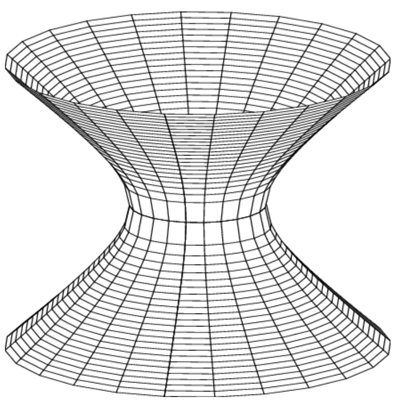


If you hang a cable or rope between two pedestals, the curve that will be formed is called a catenary. The equation of a catenary is

$$y = a\,cosh\left(\frac{x}{a}\right)$$

This is the same curve used in a suspension bridge such as the Golden gate bridge.

If you rotate this surface around a center, you get a catenoid:



This is the shape a bubble that is created between two bubbles will make.

In architecture,

- if you have a free-standing (i.e. unloaded and unsupported) arch, the optimal shape to handle the lines of thrust produced by its own weight is $\cosh(x)$. The dome of Saint Paul's Cathedral in England has a $\cosh(x)$ cross-section.