Math 109 Calc 1 Lecture 17

Linear Approximations

Section 3.10

Linear approximation is a technique you may run into in the sciences, economics or other fields. It is an applied mathematical tool that can be used when dealing with real world measurements where exact answers are not necessary.

We start with the derivative formula.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

If we relax the condition that h goes to zero, but rather just require h to be small, we can get an approximation:

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

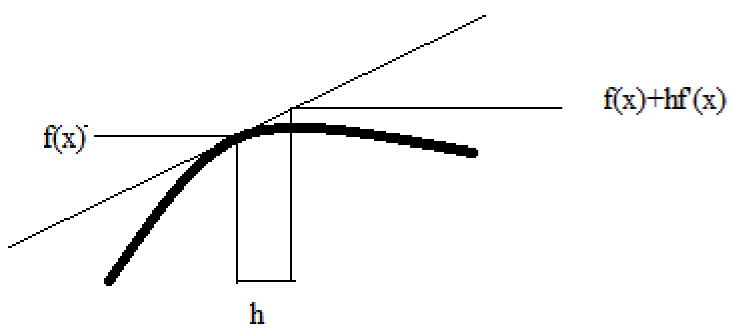
We can rearrange this as

$$hf'(x) \approx f(x+h) - f(x)$$

or

$$f(x+h) \approx f(x) + hf'(x)$$

This says that if we know $f(x)$ and $f'(x)$ then for small h we can approximate $f(x+h)$.



Example:

Let $f(x) = \sqrt{x}$, approximate $\sqrt{104}$

$$f'(x) = \frac{1}{2\sqrt{x}}$$

$$f(100+4) \approx f(100) + 4f'(100) = \sqrt{100} + \frac{4}{2\sqrt{100}} = 10 + \frac{1}{5} = 10.2$$

The actual value of $\sqrt{104} \approx 10.198$ so this approximation is $\frac{.002}{10.2} \times 100 = .02\%$

The formula for this relative percent error is

%Error = $100 \times \frac{hf\prime(x)}{f(x)}$

Example:

A sphere of radius 3 in is incorrectly measured to have a diameter of 3.1 inches.

a) Estimate the error in the volume
b) Find the relative percent error

$$V(r) = \frac{4\pi}{3} r^3$$

$$\frac{dV}{dr} = 4\pi r^2$$

$V(3+.1) - V(3) \approx V'(3)(.1) = 4\pi 3^2(.1) = 3.6\pi$, the error in volume

$$100\left[\frac{V(3+.1)-V(3)}{V(3)}\right] \approx 100\left[\frac{V'(3)(.1)}{V(3)}\right] = \frac{3.6\pi}{36\pi} = .1\%$$

In Class Problems

Use differentials to find the approximate value.

1.* $\sqrt[3]{1010}$. 3.* $\dfrac{1}{\sqrt{22}}$. 5.* $\sqrt[5]{30}$. 7.* $(33)^{3/5}$.

2. $\sqrt{125}$. 4. $\dfrac{1}{\sqrt{24}}$. 6. $(26)^{2/3}$. 8. $(33)^{-1/5}$.

9.* Using differentials, find the approximate area of a ring of inner radius r and width h. What is the exact area?

10. Show that the relative error in the volume of a cube, due to an error in measuring the edge, is approximately three times the relative error in the edge.

11. Show that the relative error in the surface area of a sphere, due to an error in estimating the radius, is approximately twice the relative error in the radius.

12. Show that the relative error in the nth root of a number is approximately $1/n$ times the relative error in the number.

13.* Find the approximate volume of a thin cylindrical sheet with open ends if the inner radius is r, the height is h, and the thickness is t.

14.* A box is to be constructed in the form of a cube to hold 1000 cu ft. Use differentials to estimate how accurately the inner edge must be made so that the volume will be correct to within 3 cu ft.

15. Use differentials to estimate for what values of x
(a)* $\sqrt{x+1} - \sqrt{x} < 0.01$.
(b) $\sqrt[4]{x+1} - \sqrt[4]{x} < 0.002$.

16.* The time of one vibration of a pendulum is given by the formula

$$t = \pi\sqrt{\frac{l}{g}},$$

where t is measured in seconds, $g = 32.2$, and l is the length of the pendulum measured in feet. Taking $\pi = 3.14$, find
(a) the length of a pendulum that vibrates once a second;
(b) the approximate change in t if the pendulum is lengthened 0.01 ft;
(c) the approximate time gained or lost per day by a pendulum clock with this error.

17. Show that the derivative $f'(x)$ is the *only* number that can satisfy the equation

$$f(x+h) - f(x) = f'(x)h + o(h).$$

[HINT: Show that, if $f(x+h) - f(x) = Nh + o(h)$, then $N = f'(x)$.]