Math 109 Calc 1 Lecture 15

Exponential Growth and Decay

Section 3.8

Many real-world variables have the property that their rate of growth or decay are a function of their value.

Here are some examples.

The population of animals or bacteria.

The human population.

Radioactive decay.

Compound interest.

We start with an abstract mathematical view of this type of phenomena.

$$\frac{dy}{dt} = ky$$

This indicates that the function $y(t)$ has a rate of change equal to the value of y at some moment in time, times some constant.

This is known as a differential equation as it includes both a function and one of its derivatives.

The study of differential equations is vast. Anyone studying Physics or Engineering will find themselves in at least an entire semester course dedicate to this subject. Chemists, biologist, astronomers and economists will also likely find themselves in such a course.

Since we have a function whose derivative is equal to the function times a constant, this causes us to consider any function we know of that might have this property. The obvious answer is

$y = e^x$ as we know that $\frac{d}{dx}e^x = e^x$

We embellish our function $y = Ce^{kt}$ and substitute it into our differential equation

$$\frac{dy}{dt} = ky$$

$$kCe^{kt} = kCe^{kt}$$

So $y = Ce^{kt}$ is a solution to our differential equation.

The variable *k* seems related to the rate of change, and the variable *C* seems currently unbounded, any *C* will work.

We note that when *t* = 0, $y = Ce^0 = C$ so *C* is the value of *y* at time *t*.

If we solve for *k* we see that

$$k = \frac{\frac{dy}{dt}}{y}$$

We have a name for this, the **relative growth rate**.

Population Growth

Example: population growth

Assume for the moment that relative growth rate of the human population has remained constant from 1950 through 2025. It is doubtful that this is exactly the case, however it will serve as a reasonable estimate.

Given that the world population was 2,560M in 1950 and 3,040M in 1960, what will the population be in 2025?

Taking 1950 as *t*=0 and therefore in 1993 *t* =10 and in 2025 *t* = 75.

Using $P(t) = Ce^{kt}$ we find that $P(0) = Ce^0 = C = 2560$

$$P(10) = Ce^{10k} = 2560e^{10k} = 3040$$

Solving for *k*

$$k = \frac{\ln \frac{3040}{2560}}{10} \approx 0.017185$$

So, we have $P(75) = 2560e^{75 \cdot 0.017185} = 9289$ Since the current population of the world is about 7753M, one can conclude that the population growth rate has slowed.

Radioactive Decay

Radioactive substances decay at a rate proportional to their mass or the number of atoms present.

For a decay problem, k = be negative.

The decay times for a substance is listed as a **half-life**.   This is the time it takes for the amount of radioactive material to decrease in half.

Example:

The half-life of radium-266 is 1590 years.

If you were to start with a kilogram of pure radium-266, after 1590 years, only half of the radium-266 atoms would be left.

A sample of radium-266 has mass 100mg.

To find a formula for the amount left after t years, note that

$R(0)=100$ and $R(1590)=50$

So, we have $R(t)=100e^{kt}$

And we know that $R(1590)=100e^{1590k}=50$

Solving for k we get $k=\ln(.5)/1590$

To find the amount of radium-266 after 100 years,

$R(1000)=100e^{1000\ln(.5)/1590}\approx 64.66$

To find when the amount of radium will decay to 30mg

$30=100e^{t\ln(.5)/1590}$ or $t=\ln\left(\frac{30}{100}\right)/\left(\ln(.5)/1590\right)\approx 2762$ years

Compound Interest

If you invest some money *P* for principle at a certain yearly interest *r* compounded yearly for *t* years, you end up with the equation for the future value

$$F(t) = P(1+r)^t$$

If instead we compound quarterly, the equation changes to

$$F(t) = P\left(1+\frac{r}{4}\right)^{4t}$$

So, if you compound *n* times a year the equation becomes

$$F(t) = P\left(1+\frac{r}{n}\right)^{nt} = P\left[\left(1+\frac{r}{n}\right)^{n/r}\right]^{rt}$$

We substitute *m*=*n/*r

$$F(t) = P\left[\left(1+\frac{1}{m}\right)^{m}\right]^{rt}$$

And we take the limit as *m* goes to infinity, which is the same as *n* going to infinity

$$F(t) = \lim_{m\to\infty} P\left[\left(1+\frac{1}{m}\right)^{m}\right]^{rt} = P\left[\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{m}\right]^{rt}$$

We examine this limit for large *m*

$$\left(1+\frac{1}{10}\right)^{10} = 2.5937$$

$$\left(1+\frac{1}{100}\right)^{100} = 2.7048$$

$$\left(1+\frac{1}{1000}\right)^{1000} = 2.7169$$

$$\left(1+\frac{1}{10000}\right)^{10000} = 2.71814$$

We can see that this number has a limit of *e*, the Euler constant.

So, our formula for future value when we compound continuously is

$$F(t) = Pe^{rt}$$

Example:

If you invest $5000 at %2 interest for 3 years, what is your future value?

$$F(t) = Pe^{rt} = 5000e^{(.02)(3)} = \$5309.18$$