Math 109 Calc 1 Lecture 14

Inverse Function Rule

Section 3.6

Inverse Functions

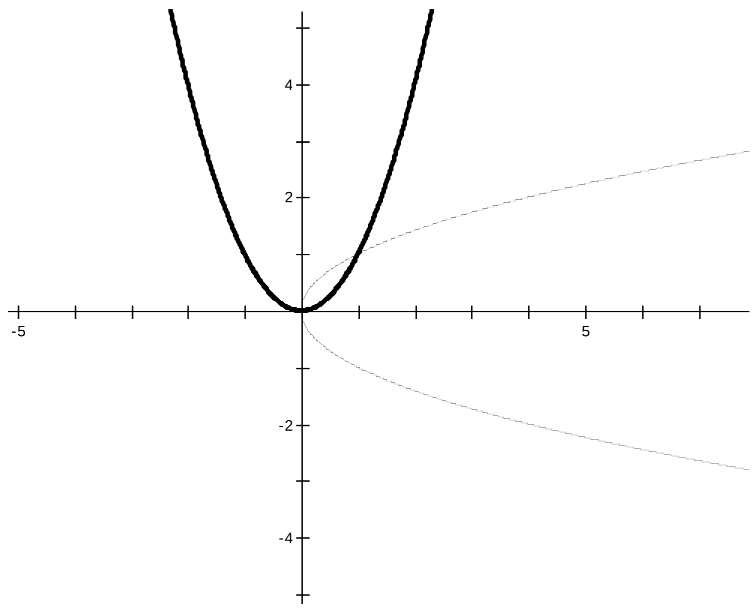
Recall that an inverse function takes the output of another function as input and returns the original input.

Example:

If $f(x) = 2x + 1$ then $f^{-1}(x) = \frac{x-1}{2}$

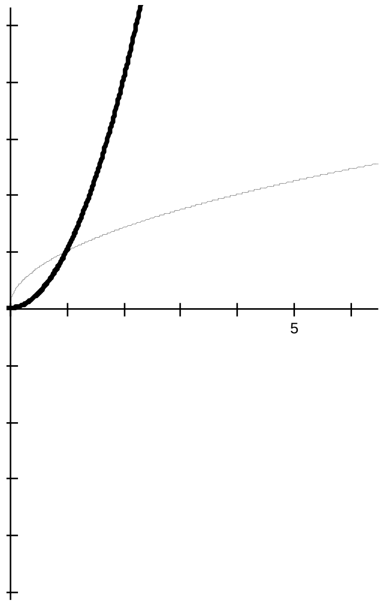
The inverse of some functions is not a function, for example if

If $f(x) = x^2$ the its inverse is $\pm\sqrt{x}$



However, if we suitably constrain the domain of a function, we can often make its inverse a function.

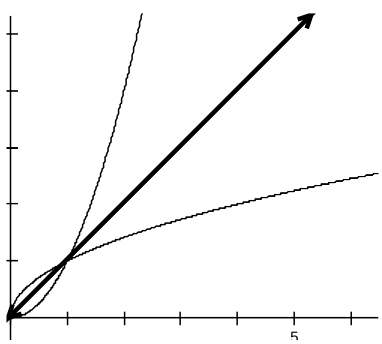
If $f(x) = x^2$ with domain $[0, \infty)$ then its inverse is $\sqrt{x}$



Inverse functions have the properties that

$$f\left(f^{-1}(x)\right) = f^{-1}\left(f(x)\right) = x$$

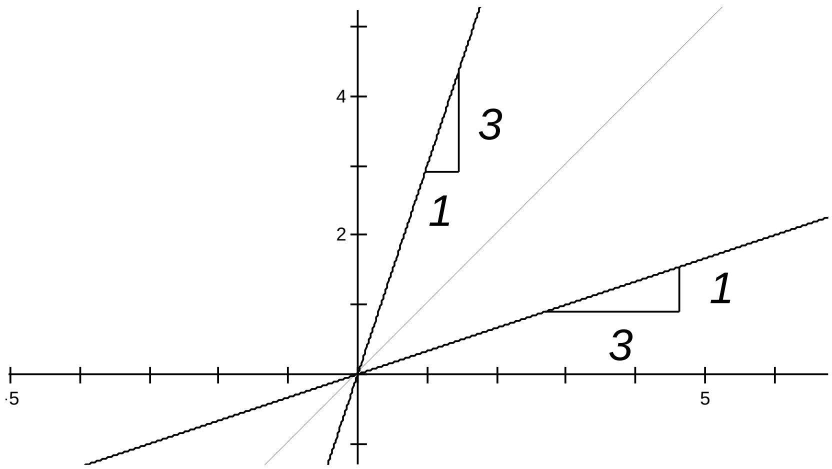
It is also worth noting in the diagram that the graph of an inverse function is a reflection of the function in the line $y = x$.



If we don't know the derivative of the inverse of a function, we would like to find it using the derivative of the original function.   First let's look at a simple example.

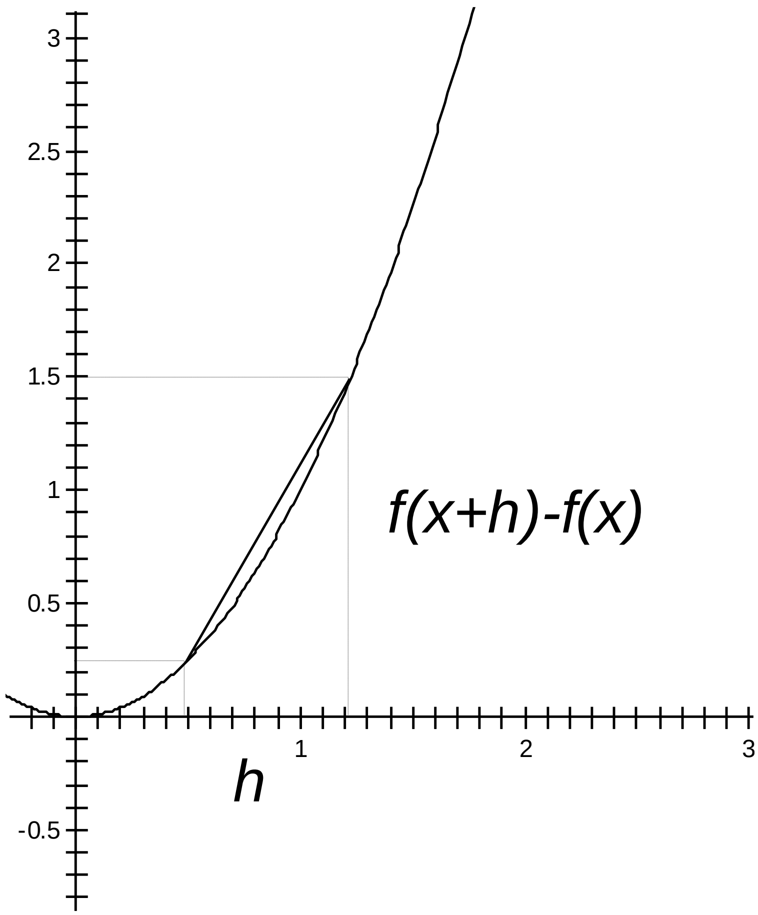
$$f(x) = 3x$$

$$f^{-1}(x) = \frac{1}{3}x$$



Note that the $\frac{\Delta y}{\Delta x}$ gets reversed to $\frac{\Delta x}{\Delta y}$

Looking at the derivative of a function



$$f'(x) = \lim_{x \to 0} \frac{f(x+h) - f(x)}{h}$$

If we switch the *x* and *y* axes, this becomes

$$\left[f^{-1}(x) \right]' = \lim_{x \to 0} \frac{h}{f(x+h) - f(x)} = \frac{1}{f'(x)}$$

So, the derivative of the inverse of a function is the reciprocal of the derivative of the function.

$$\left(f^{-1}(x)\right)' = \frac{1}{f'(x)}$$

or in Leibniz notation

$$\frac{dx}{dy} = 1 \Big/ \frac{dy}{dx}$$

This should seem plausible since reflecting a tangent line across *y* = *x* switches the *x* and *y* coordinates and so the slope $m = \frac{\Delta y}{\Delta x}$ becomes $\frac{\Delta \mathrm{x}}{\Delta \mathrm{y}}$

Let's check this with the function $y = x^2$

We know that $y' = 2x$

$$\frac{dx}{dy} = 1/\frac{dy}{dx} = \frac{1}{2x} = \frac{1}{2}x^{-1}$$

Since $x = \sqrt{y}$ we have $\frac{dx}{dy} = \frac{1}{2\sqrt{y}}$ Which is what we would have expected using the power rule.

We next want to try this on some new ground.

We investigate the derivative of the function $y = \log_e x$ by using the derivative of its inverse, the function $y = e^x$

We know that $y' = e^x$

$$\frac{dx}{dy} = 1/\frac{dy}{dx} = \frac{1}{e^x} = \frac{1}{y}$$

So, we conclude that $\frac{d}{dx}\log_e x = \frac{1}{x}$

The Inverse Trig Functions

For $y = sin(x)$ and $y' = cos(x)$

$$\frac{dx}{dy} = 1/\frac{dy}{dx} = \frac{1}{cos(x)} = \frac{1}{\sqrt{1 - sin^2(x)}} = \frac{1}{\sqrt{1 - y^2}}$$

So, we have that

$$\frac{d}{dy} sin^{-1}(x) = \frac{1}{\sqrt{1 - x^2}}$$

We can also find that

$$\frac{d}{dy} cos^{-1}(x) = -\frac{1}{\sqrt{1 - x^2}}$$

For $y = tan(x)$ and $y' = sec^2(x)$

$$\frac{dx}{dy} = 1/\frac{dy}{dx} = \frac{1}{sec^2(x)}$$

At this point we have to recall a well-known trig identity.

Since, $sin^2(x) + cos^2(x) = 1$

$$\frac{sin^2(x)}{cos^2(x)} + \frac{cos^2(x)}{cos^2(x)} = \frac{1}{cos^2(x)}$$

Therefore

$$tan^2(x) + 1 = sec^2(x)$$

$$\frac{dx}{dy} = \frac{1}{tan^2(x) + 1} = \frac{1}{y^2 + 1}$$

$$\frac{d}{dy} tan^{-1}(x) = \frac{1}{x^2 + 1}$$

Logarithmic Differentiation

Combining implicit differentiation and our knowledge of the derivative of the natural log we have a method to simplify certain derivative problems.

Example

$$y = \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$$

As it stands this looks like a lot of work, combining the power, product and quotient rules. But here we take a different approach by taking the log of both sides and using implicit differentiation.

$$\log_e y = \log_e \frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5} = \log_e x^{3/4}\sqrt{x^2+1} - \log_e(3x+2)^5 =$$

$$\frac{3}{4}\log_e x \quad + \frac{1}{2}\log_e(x^2+1) - 5\log_e(3x+2)$$

Taking the log of both sides we get

$$\frac{1}{y}y' = \frac{3}{4x} + \frac{2x}{2(x^2+1)} - 5\frac{3}{3x+2} = \frac{3}{4x} + \frac{x}{(x^2+1)} - \frac{15}{3x+2}$$

Multiplying by *y*

$$y' = \left(\frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}\right)\left(\frac{3}{4x} + \frac{x}{(x^2+1)} - \frac{15}{3x+2}\right)$$

While this save some time and effort, it was still possible to do without logarithmic differentiation.   Next is an example where this is not the case.

$$y = x^x$$

$$\log_e y = \log_e x^x = x \log_e x$$

Finding the derivative of both sides we get

$$\frac{1}{y} y' = (x \log_e x)' = x \cdot \frac{1}{x} + \log_e x = 1 + \log_e x$$

Multiplying both sides by y

$$y' = x^x (1 + \log_e x)$$