Math 109 Calc 1 Lecture 12

The Chain Rule

The Chain Rule (Section 3.4)

We're now going to look at

Section 3.4 the Chain Rule

In algebra we come across the use of composition of functions, for example

If $f(x) = \sqrt{x}$ and $g(x) = x^2 + 1$ we can have the composition of these functions written as either

$(f \circ g)(x)$ or $f\big(g(x)\big) = \sqrt{x^2 + 1}$

Currently we don't have a way to find the derivative of such a function.

There are many circumstances when trying to find a derivative where it makes sense to view the function as a composition of functions.

The rule that allows us to figure out the derivative of a composed function is called the chain rule. We'll see why it gets this name later.

The chain rule is usually written in one of two ways

In Newtonian syntax we write $\big((f \circ g)(x)\big)' = [f(g(x)]' = f'(g(x))g'(x)$

We would say this as "The derivative of *f* of *g* of *x* is the derivative of *f* at *g*(*x*) times the derivative of *g* at x".

With Leibniz notation we write this as $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$ where we have $y = f(u)$ and $u = g(x)$.

How plausible is this?

Let's take two simple examples.

Example:

$f(x) = x$
$g(x) = x^2$

$f'\big(g(x)\big) = 1$
$g'(x)$=2x
$f'\big(g(x)\big)g'(x) = 2x$

$h(x) = f(g(x)) \; = \; x^2$
$h'(x) \; = \; 2x$

Example:

$f(x) = x^2$
$g(x) = x^2$

$f'\big(g(x)\big) = 2(x^2) = 2x^2$
$g'(x)$=2x
$f'\big(g(x)\big)g(x) = 4x^3$

$h(x) = f(g(x)) \; = \; x^4$
$h'(x) \; = 4x^3$

Example:

If $f(x) = \sqrt{x}$ and $g(x) = x^2 + 1$

$$f\big(g(x)\big)' = f'(g(x) \cdot g'(x) \; = \; \frac{-1}{2\sqrt{x^2+1}} \cdot 2x$$

If $f(x) = e^x$ and $g(x) = x^3$

$$f\big(g(x)\big)' = \left[e^{x^3}\right]' = e^{x^3} \cdot 3x^2$$

Example:

If $f(x) = \sqrt{x}$ and $g(x) = x^2 + 1$

$$f\left(g(x)\right)' = \left[\sqrt{x^2 + 1}\right]' = \frac{1}{2\sqrt{x^2 + 1}} \cdot 2x = \frac{\mathrm{x}}{\sqrt{x^2 + 1}}$$

Example:

If $f(x) = x^2$ and $g(x) = sin(x)$

$$f\left(g(x)\right)' = [sin^2(x)]' = 2(sin(x)) \cdot cos(x)$$

Example:

If $f(x) = sin(x)$ and $g(x) = x^2$

$$f\left(g\left(x\right)\right)' = \left[\sin\left(x^2\right)\right]' = \cos\left(x^2\right) \cdot 2x$$

Why a chain?

Suppose you need to find the derivative of the function

$$f(x) = e^{\sqrt{x^3+2x+1}}$$

You can look at this at the composition of three functions

It's easier to look at this using the Leibniz notation

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dv}\frac{dv}{dx}$$

where

$y = e^u$
$u = \sqrt{v}$
v = $x^3 + 2x + 1$

So, we have

$$\frac{dy}{du} = e^u$$

$$\frac{du}{dv} = \frac{-1}{2\sqrt{v}}$$

$$\frac{dv}{dx} = 3x^2 + 2$$

Putting this together we get

$$y' = e^{\sqrt{x^3+2x+1}}\left(\frac{-1}{2\sqrt{x^3+2x+1}}\right)(3x^2+2)$$

Example:

$$f(x) = e^{sin^2(3x)}$$

Example:

$$f(x) = (e^{3x+2} + 5x^2)^5$$

Example:
Using Leibniz notation

If $y = 3u^2$+2 and $u = \frac{1}{x-1}$

$$\frac{dy}{du} = 6u$$

$$\frac{du}{dx} = \frac{-1}{(x-1)^2}$$

$$\frac{dy}{du}\frac{du}{dx} = 6u\frac{-1}{(x-1)^2} = 6\left(\frac{1}{x-1}\right)\frac{-1}{(x-1)^2} = \frac{-6}{(x-1)^3}$$

Some In Class Examples:

1.* $f(x) = (1 - 2x)^{-1}$.

2. $f(x) = (1 - 2x)^5$.

3.* $f(x) = (x^5 - x^{10})^{20}$.

4. $f(x) = \left(x^2 - \frac{1}{x^2}\right)^3$.

5.* $f(x) = x - \frac{1}{x}$.

6. $f(x) = \left(x - \frac{1}{x}\right)^3$.

7.* $f(x) = (x - x^3 - x^5)^4$.

8. $f(t) = \left(\frac{1}{1+t}\right)^4$.

9.* $f(t) = (t^2 - 1)^{100}$

10. $f(t) = (6 + t^2)^3$

11.* $f(t) = (t^{-1} + t^{-2})^4$

12. $f(x) = \left(\frac{ax+b}{cx+d}\right)^{-3}$.

13.* $f(x) = \left(\frac{3x}{x^2+1}\right)^4$.

14. $f(x) = [(2x+1)^2 - (x+1)^2]^3$.

15.* $f(x) = (x^4 + x^2 + x)^2$.

16. $f(x) = (x^2 + x + 1)^3$.

17.* $f(x) = \left(\frac{x^3}{3} + \frac{x^2}{2} + \frac{x}{1}\right)^{-1}$.

18. $f(x) = \left(\frac{x^2-1}{x^2+1}\right)^5$.

19.* $f(x) = \left(\frac{1}{x+2} - \frac{1}{x-2}\right)^3$.

20. $f(x) = [(6x - x^5)^{-1} + x]^2$.

Find dy/dx.

21.* $y = \frac{1}{1+u^2}$, $u = 2x + 1$.

22. $y = u + \frac{1}{u}$, $u = (3x+1)^4$.

23.* $y = \frac{2u}{1-4u}$, $u = (5x^2+1)^4$.

24. $y = u^3 + u + 1$, $u = \frac{1-x}{1+x}$.

Find dy/dt.

25. $y = \frac{1-7u}{1+u^2}$, $u = 1 - x^2$, $x = 2t - 5$.

26. $y = 1 - u^2$, $u = \frac{1-7x}{1+x^2}$, $x = 5t - 2$.