Vectors Mathematics 108

Vectors

A **vector** is a special type of mathematical object.

In science, especially Physics, we use a particular kind of vector,
a spatial object that both a magnitude and a direction.

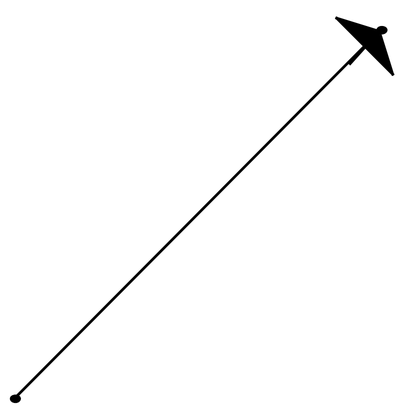
Example of quantities that can be represented by these vectors include velocity and acceleration. To describe either of these quantities, you need to describe both a direction and a magnitude.

Physics makes a distinction between the vector quantity velocity and term **speed**.
Speed is just a measure of the magnitude of a velocity vector, but without reference to the direction. The magnitude of a vector is always positive.

So for example, I could say my speed when driving to LA was 65mph.
However when I was driving to LA my velocity was 65mph south.

Physicists will refer to a number such as speed as a **scalar** quantity.

One way to represent a vector graphically is as a directed line segment.



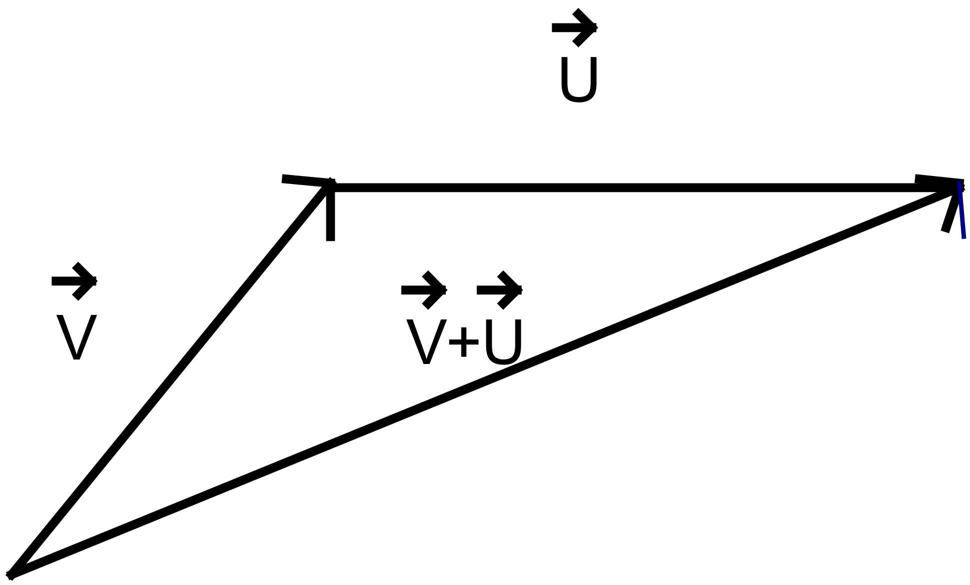
Note that we place an arrowhead on one end of the segment to indicate the direction.
The length of the segment is used to represent the vectors magnitude.

We call the point without the arrowhead the **tail** of a vector and the point with the arrowhead, the **head**.

We use can label a vector using a letter, often *v* but with an arrow on top.
$\vec{v}$

Adding Vectors

If we are using a graphical representation, we can add vectors by moving the head of one vector to the tail of the other. The vector representing the sum is then represented by a vector which starts at the open tail and goes to the open head.



A vector can also be represented algebraically by it's coordinates.

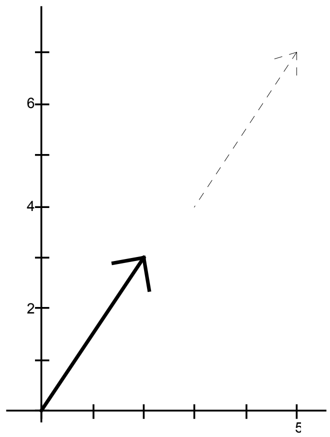
For example a vector my have a tail at (3,4) and a head at (5,7)

(3,4) ---> (5,7)

Only the direction and magnitude of the vector are important so a vector can be moved as follows without changing it.

(3-*x*,4-*y*) ---> (5-*x*,7-*y*)

It is common to move a vector so its tail is at the origin.



This is called a **position vector**.

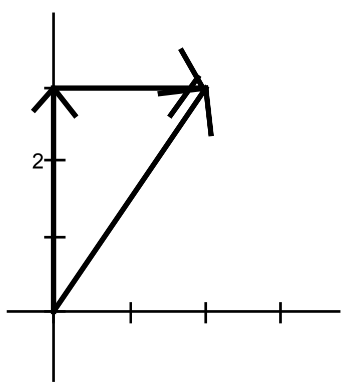
(3-3,4-4) --->(5-3,7-4)

(0,0)--->(2,3)

We can write this type of vector using angle brackets in **component form**.

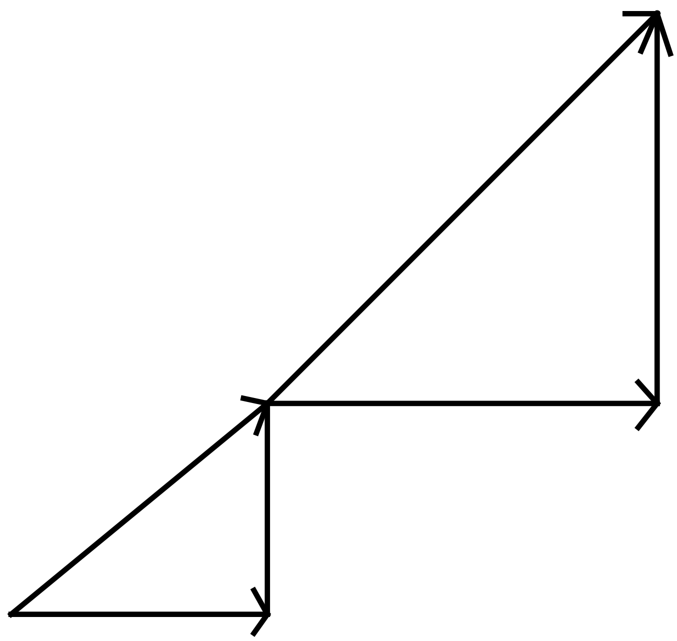
$\vec{v} = \langle 2,3 \rangle$

This name comes from the fact that 2 and 3 are the magnitudes of vectors in the *y* and *x* direction that add up to this vector.



One convenient feature of the component form is that it makes it very easy to add two vectors by just adding their components.

$$\vec{v} = \langle a, b \rangle$$
$$\vec{u} = \langle c, d \rangle$$
$$\vec{v} + \vec{u} = \langle a + c, b + d \rangle$$



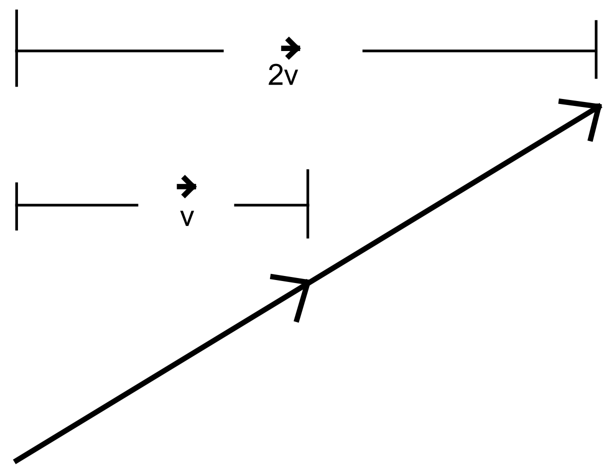
Another nice feature of component form is that it is easy to calculate the magnitude of a vector using the Pythagorean theorem.

$$\vec{v} = \langle a, b \rangle$$
$$\left|\vec{v}\right| = \sqrt{a^2 + b^2}$$

This is also known as the **norm** of a vector.

Multiplying Vectors by a Scalar

A vector can be multiplied by a number, or a scalar. This does not change the direction of the vector, but either stretches it or shortens the vector.



This is also quite easy in component form

$$\vec{v} = \langle a, b \rangle$$
$$k\vec{v} = \langle ka, kb \rangle$$

This brings up a question. What is

$$0\vec{v} = \langle 0, 0 \rangle = \vec{0}$$

This is the **zero vector**.
The zero vector has magnitude zero, and has no direction, or any direction.

Like the number zero, if you add it to a vector, you just get the vector back, so it has the identity property

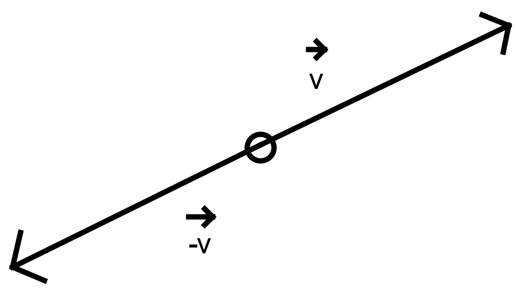
$$\vec{v} + \vec{0} = \vec{v}$$

Subtracting Vectors

Since we can add vectors, we should be able to subtract them. We do this by adding a vector in the opposite direction.

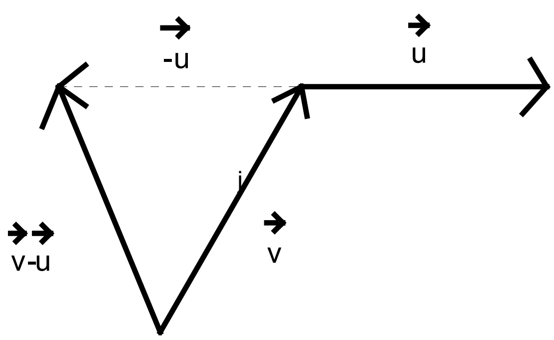
Note that multiplying a vector by -1 just reverses its direction.

$$\vec{v} = \langle a, b \rangle$$
$$-1 \cdot \vec{v} = -\vec{v} \langle -a, -b \rangle$$

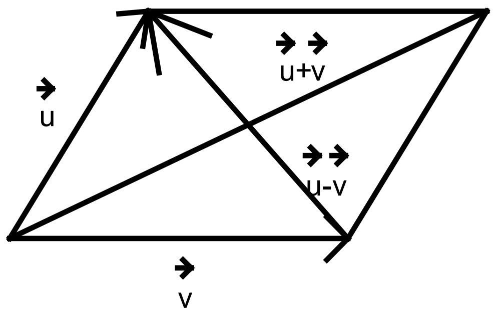


So to subtract a vector we just add its opposite.

$$\vec{v} = \langle a, b \rangle$$
$$\vec{u} = \langle c, d \rangle$$
$$\vec{v} - \vec{u} = \vec{v} + -\vec{u} \langle a - c, b - d \rangle$$



Notice that when you add two vectors that are not in the same direction or opposite direction, there is a parallelogram formed. The sum of the vectors is one of the diagonals and the difference is the other.



The properties of vectors are similar to numbers so it's worth enumerating them.

Vector Addition

$\vec{v}+\vec{u}=\overrightarrow{v+u}$	Closure
$\vec{v}+\vec{u}=\vec{u}+\vec{v}$	Commutative
$\left(\vec{v}+\vec{u}\right)+\vec{w}=\vec{v}+\left(\vec{u}+\vec{w}\right)$	Associative
$\vec{v}+\vec{0}=\vec{v}$	Identity
$\vec{v}+\overrightarrow{-v}=\vec{0}$	Inverse

Scalar Multiplication

$c\left(\vec{v}+\vec{u}\right)=c\vec{v}+c\vec{u}$	Distributive
$\left(c+d\right)\vec{v}=c\vec{v}+d\vec{v}$	Distributive
$1\cdot\vec{v}=\vec{v}$	Identity
$0\cdot\vec{v}=\vec{0}$	Zero
$c\vec{0}=\vec{0}$	Zero

Property of the Norm

$$\left|c\vec{v}\right|=\left|c\right|\cdot\left|\vec{v}\right|$$

Note that there are two meanings of || here, norm and absolute value.

Unit Vectors

If the norm or magnitude of a vector is 1, it is called a unit vector.

Example:

$$\left|\left\langle \frac{3}{4}, \frac{4}{5} \right\rangle\right| = \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = \sqrt{\frac{9}{25} + \frac{16}{25}} = \sqrt{\frac{25}{25}} = \sqrt{1} = 1$$

Two very important unit vectors are given the names $\vec{i}$ and $\vec{j}$ are

$$\vec{i} = \langle 1, 0 \rangle$$
$$\vec{j} = \langle 0, 1 \rangle$$

Note that if we have a vector

$\vec{v} = \langle a, b \rangle$ we can represent it as a **linear combination** of $\vec{i}$ and $\vec{j}$.

$$\langle a, b \rangle = a\vec{i} + b\vec{j}$$

Using Trigonometry to find the direction of a vector

Note that for

$$\vec{v} = \langle a, b \rangle$$
$$\theta = \arctan\left(\frac{b}{a}\right)$$

Of course we need to adjust the quadrant depending on the signs of *a*, and *b*.

Also, given the direction and magnitude of a vector, we can find the components using the formulas

$$\langle a, b \rangle$$
$$a = M\cos\theta$$
$$b = M\sin\theta$$

So we have

$$\vec{v} = \left(\left|\vec{v}\right|\cos\theta\right)\vec{i} + \left(\left|\vec{v}\right|\cos\theta\right)\vec{j}$$