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Euler's Equation

What we've learned about complex numbers so far suggests a question,

what is A^z where *A* is a real number and *z* is complex?

It's not even clear yet, what it means.

Assuming the laws of exponents must work with this we have
$A^{a+ib} = A^a A^{ib}$

Since *a* and *b* are real numbers we know that A^a is a real number, so we only need to know what A^{ib} is.

Let's start with the function

$$f(\theta) = \cos\theta + i\sin\theta$$

Note that

$$f(\theta_1) f(\theta_2) = (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) =$$
$$\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2) = f(\theta_1 + \theta_2)$$

so

$$f(\theta_1) f(\theta_2) = f(\theta_1 + \theta_2)$$

This suggests that if we assume that $A^{i\theta} = f(\theta)$ it will follow the law of exponents that
$A^{i\theta_1} A^{i\theta_2} = A^{i\theta_1 + i\theta_2}$

What's left is figure out what our *A* is.

The discovery is attributed to Leonard Euler, whom the constant e is named after.
The surprising answer is that

$$e^{i\theta} = \cos\theta + i\sin\theta$$

This is known as Euler's formula.

There are a few different ways to confirm this number, however they all rely on calculus.

By substituting $\theta = \pi$ you get

$$e^{i\pi} = \cos\pi + i\sin\pi = -1$$

$e^{i\pi} = -1$ is known as Euler's identity and is considered one of the most famous equations in mathematics.