Trigonometry 16 Mathematics 108

Complex Numbers in Polar Coordinates

We are going to apply what we know about polar coordinates to the complex number plane.

We will use the variable z to mean a complex number, so we have

$z = a + bi$ where a and b are real numbers.

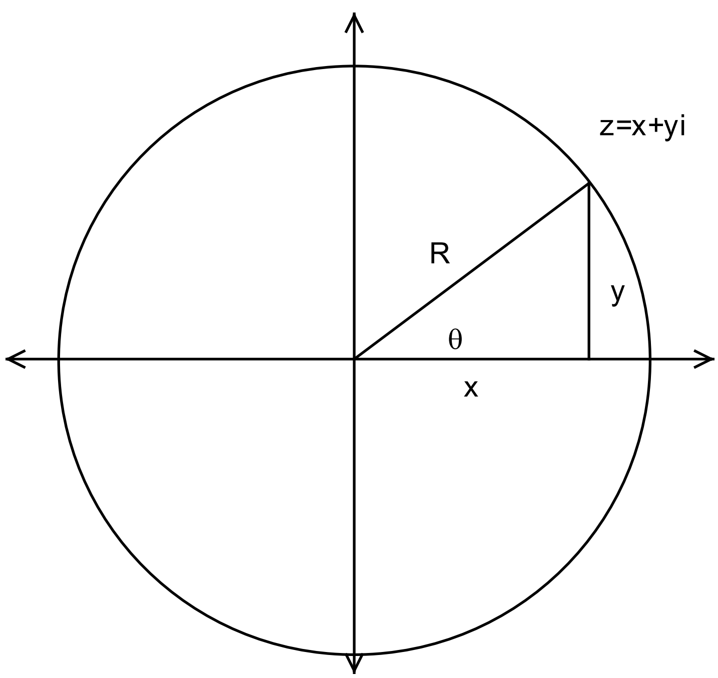
We write the complex conjugate of z
$z^* = a - bi$

If multiply a complex number times it's conjugate we get

$zz^* = (a+bi)(a-bi) = a^2 + b^2$

We define the **modulus** of z as

$|z| = \sqrt{zz^*} = \sqrt{a^2 + b^2}$
From this diagram



We see that $|z| = \sqrt{x^2 + y^2} = R$

We can then rewrite the value of z as

$$z = R\left(\frac{x}{R} + i\frac{y}{R}\right)$$

But clearly $\frac{x}{R} = \cos\theta$ and $\frac{y}{R} = \sin\theta$

So finally we can write

$$z = R(\cos\theta + i\sin\theta)$$

This is the polar form for a complex number. That is

$$x + yi = R(\cos\theta + i\sin\theta)$$

Example:

Put $z = 3 + 4i$ into polar form

$$R = |z| = \sqrt{zz^*} = \sqrt{(3+4i)(3-4i)} = \sqrt{9+16} = 5$$

so

Using our polar conversion we see that

$$\theta = \arctan\left(\frac{4}{3}\right) \approx .927$$

Since this is in the first quadrant, the sign is correct.

So we have

$$z = 5(\cos(.927) + i\sin(.927))$$

Converting back to the form $a + bi$ is obviously straight forward.

Multiplying and Dividing complex numbers in polar form

Let's look at what happens when we multiply two complex numbers in polar form.

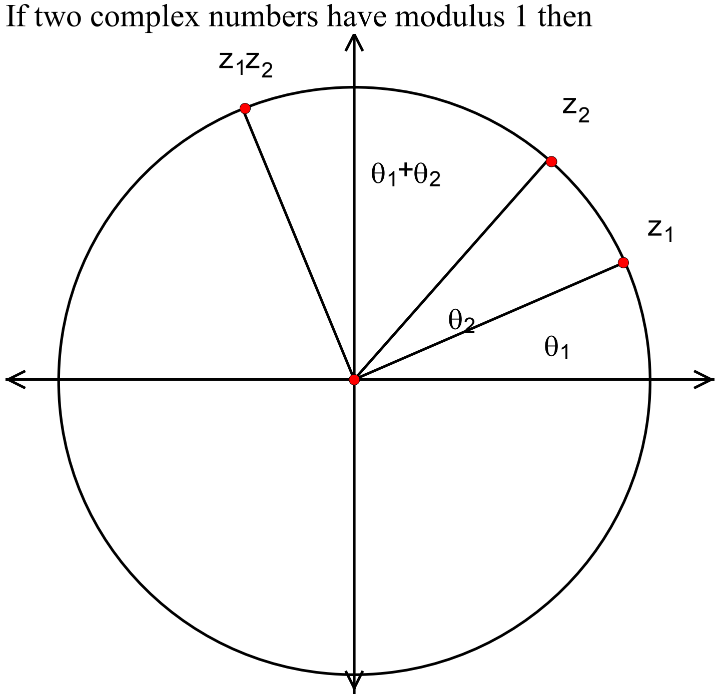
Take the numbers at coordinates (R_1, θ_1) and (R_2, θ_2) and multiply them.

$$
\begin{aligned}
&R_1(\cos\theta_1 + i\sin\theta_1)\cdot R_2(\cos\theta_2 + i\sin\theta_2) = \\
&R_1R_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) = \\
&R_1R_2[\cos\theta_1\cos\theta_2 + i\sin\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 - \sin\theta_1\sin\theta_2] = \\
&R_1R_2[\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)] = \\
&R_1R_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]
\end{aligned}
$$

That's quite amazing.

What it says is that when you multiply two complex numbers, you have to multiply their moduli which are real numbers, but you just add their angles.

If two complex numbers have modulus 1 then



Example:

$$z_1 = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$
$$z_2 = 5\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$
$$z_1 z_2 = 10\left(\cos\left(\frac{\pi}{3} + \frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{3} + \frac{\pi}{4}\right)\right) = 10\left(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right)$$

Similarly we can show for division that

$$\frac{R_1\left(\cos\theta_1 + i\sin\theta_1\right)}{R_2\left(\cos\theta_2 + i\sin\theta_2\right)} = R_1 R_2\left[\cos\left(\theta_1 - \theta_2\right) + i\sin\left(\theta_1 - \theta_2\right)\right]$$

De Moivre's Theorem

Note that if we multiply

$z^2 = R(\cos\theta + i\sin\theta)\cdot R(\cos\theta + i\sin\theta)$ we get
$z^2 = R^2(\cos 2\theta + i\sin 2\theta)$

This suggests that

$z^n = R^n(\cos n\theta + i\sin n\theta)$ for all positive integers n.

Which can be proved using induction.

Example:

Let $z = \frac{1}{2} + \frac{1}{2}i$ and find z^{10}

First put z in polar form

$|z| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 +} = \frac{1}{\sqrt{2}}$
$\theta = \arctan(1) = \frac{\pi}{4}$ so
$z = \frac{1}{\sqrt{2}}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$z^{10} = \left(\frac{1}{\sqrt{2}}\right)^{10}\left(\cos\frac{10\pi}{4} + i\sin\frac{10\pi}{4}\right) =$
$\frac{1}{32}\left(\cos\frac{5\pi}{2} + i\sin\cos\frac{5\pi}{2}\right) =$
$\frac{1}{32}\left(\cos\frac{\pi}{2} + i\sin\cos\frac{\pi}{2}\right) = \frac{1}{32}i$

We can also see that

$$\left[R^{1/n}\left(\cos\frac{\theta}{n}+i\sin\frac{\theta}{n}\right)\right]^n = R\left(\cos\theta+i\sin\theta\right)$$

So this also works for n a fraction.

So we can find complex roots of a number.

Example:

Start with the equation $x^6=-1$. This should have 6 roots, so let's find them.

$$z = 1\cdot\left(\cos\left(\pi+2\pi n\right)+i\sin\left(\pi+2\pi n\right)\right)$$

$$z^{1/6} = 1^{1/6}\cdot\left(\cos\left(\frac{\pi+2\pi n}{6}\right)+i\sin\left(\frac{\pi+2\pi n}{6}\right)\right) =$$

$$\cos\left(\frac{\pi}{6}\right)+i\sin\left(\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}+i\frac{1}{2}$$

$$\cos\left(\frac{\pi}{2}\right)+i\sin\left(\frac{\pi}{2}\right)=i$$

$$\cos\left(\frac{5\pi}{6}\right)+i\sin\left(\frac{5\pi}{6}\right)=-\frac{\sqrt{3}}{2}+i\frac{1}{2}$$

$$\cos\left(\frac{7\pi}{6}\right)+i\sin\left(\frac{7\pi}{6}\right)=-\frac{\sqrt{3}}{2}-i\frac{1}{2}$$

$$\cos\left(\frac{3\pi}{2}\right)+i\sin\left(\frac{3\pi}{2}\right)=-i$$

$$\cos\left(\frac{11\pi}{6}\right)+i\sin\left(\frac{11\pi}{6}\right)=\frac{\sqrt{3}}{2}-i\frac{1}{2}$$

So the 6 roots are $\pm i, \quad \pm\frac{\sqrt{3}}{2}\pm\frac{1}{2}$

HW: 8.4 29, 30, 31, 49, 50, 65, 66, 83, 85