Addition Subtraction Formulas

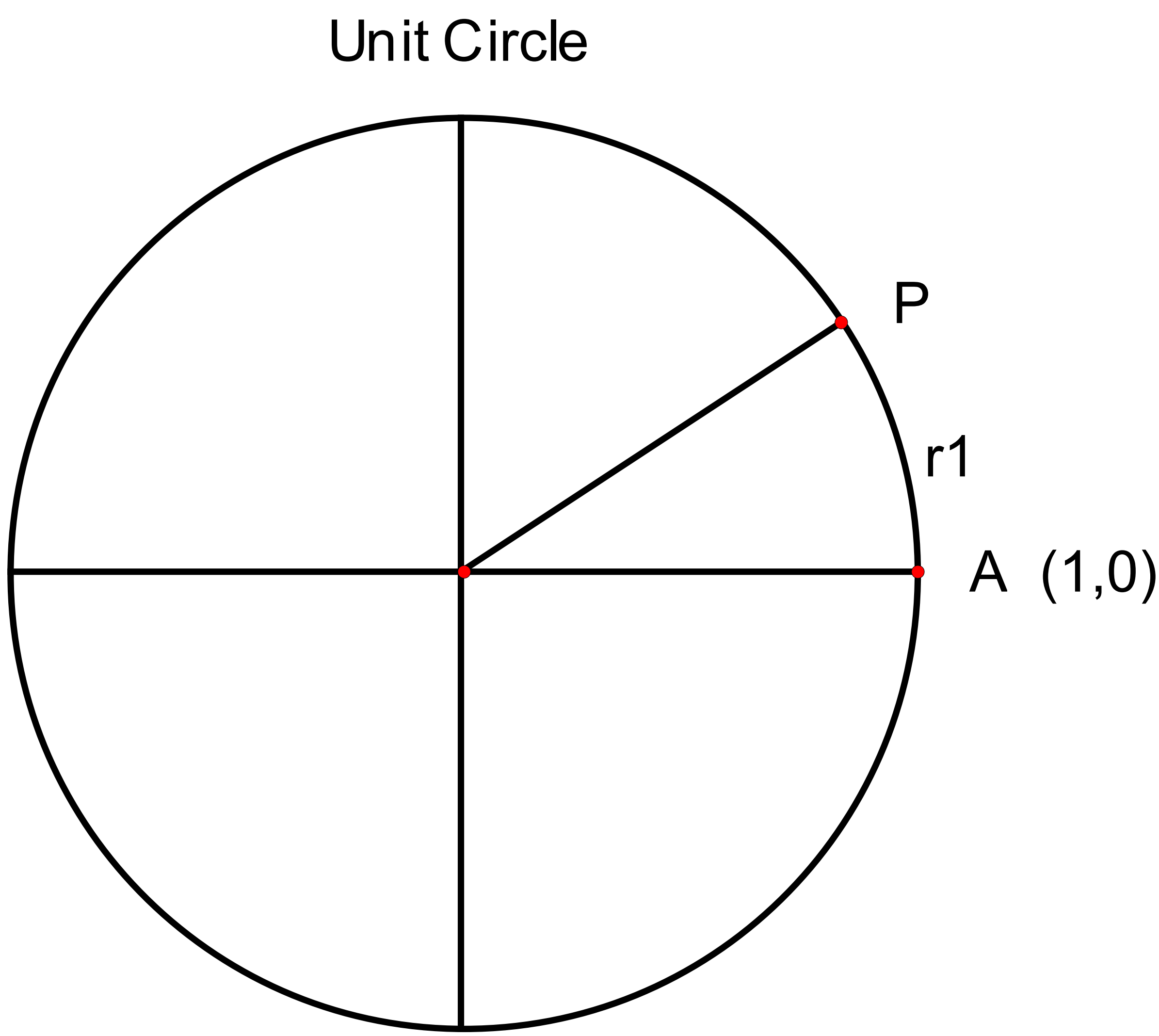
We're going to search for a summation formulae for the cosine of the sum of two angles:

$$\cos\left(x+y\right)=?$$

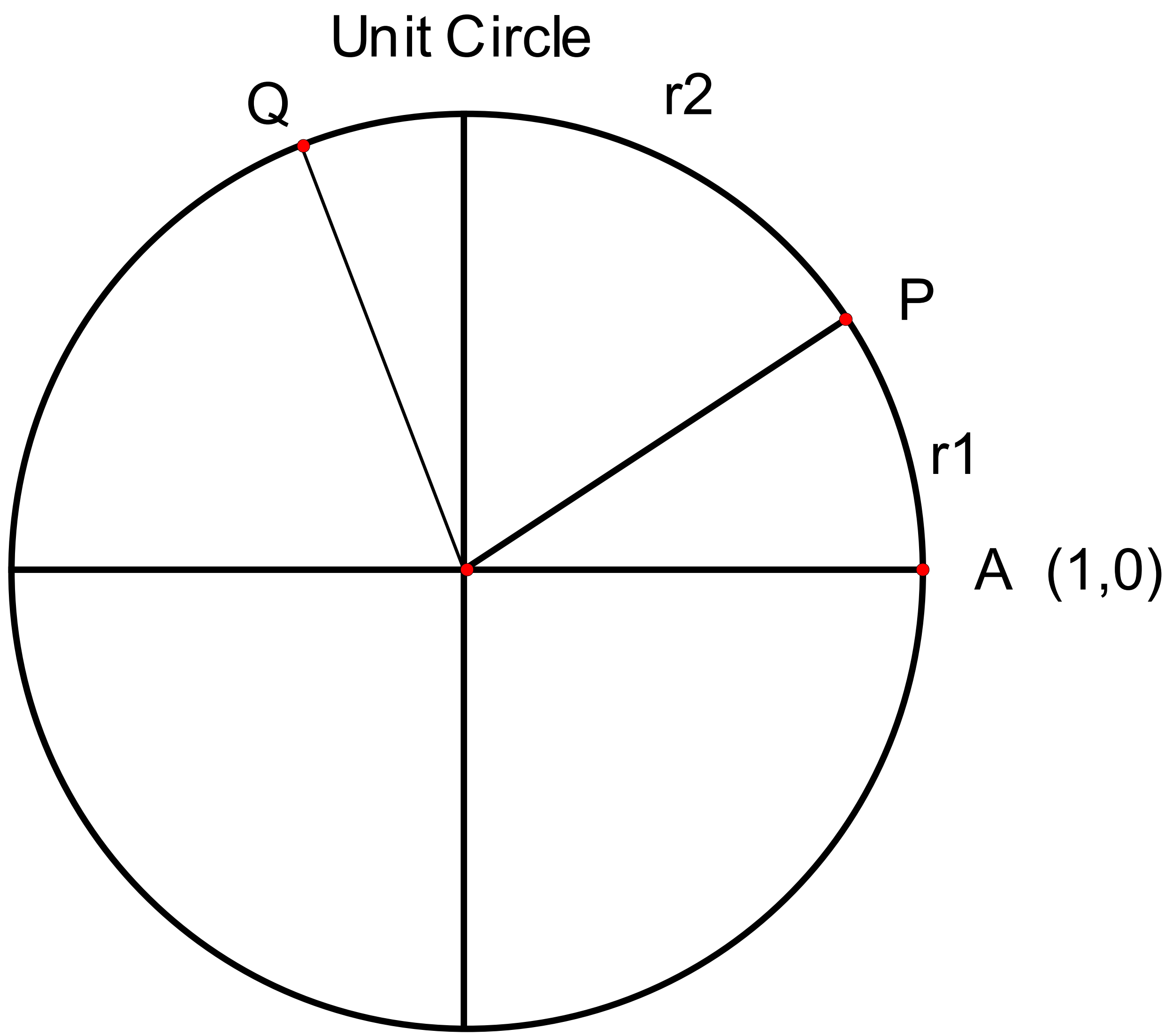
The proof we will show is a little obscure. You are not responsible for the steps, however you will need to know the result.

We start with a unit circle with point A at the coordinates (1,0) and point P an arbitrary point in the first quadrant.

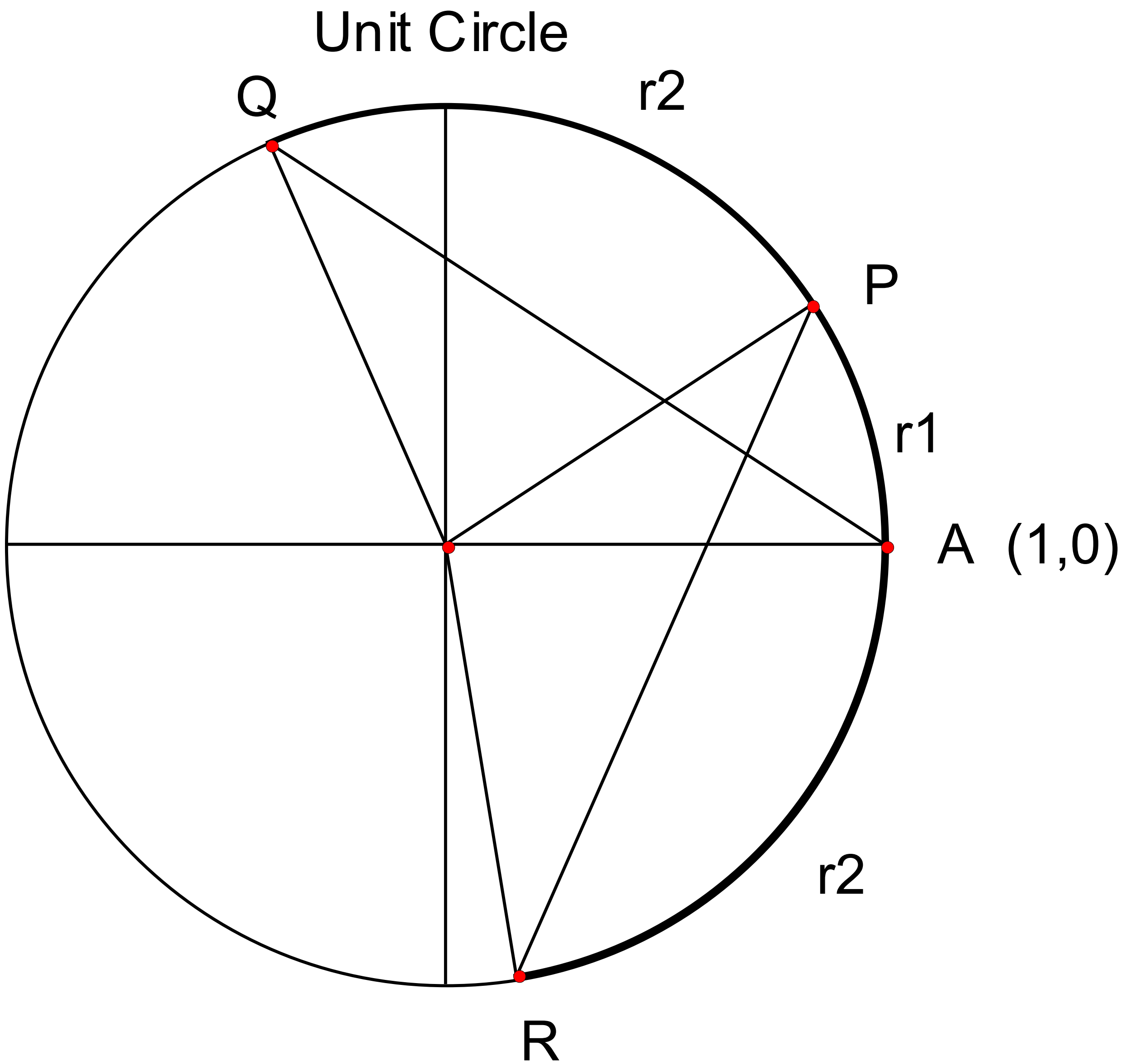
We label the angular size of arc $\widehat{AP}$ r1



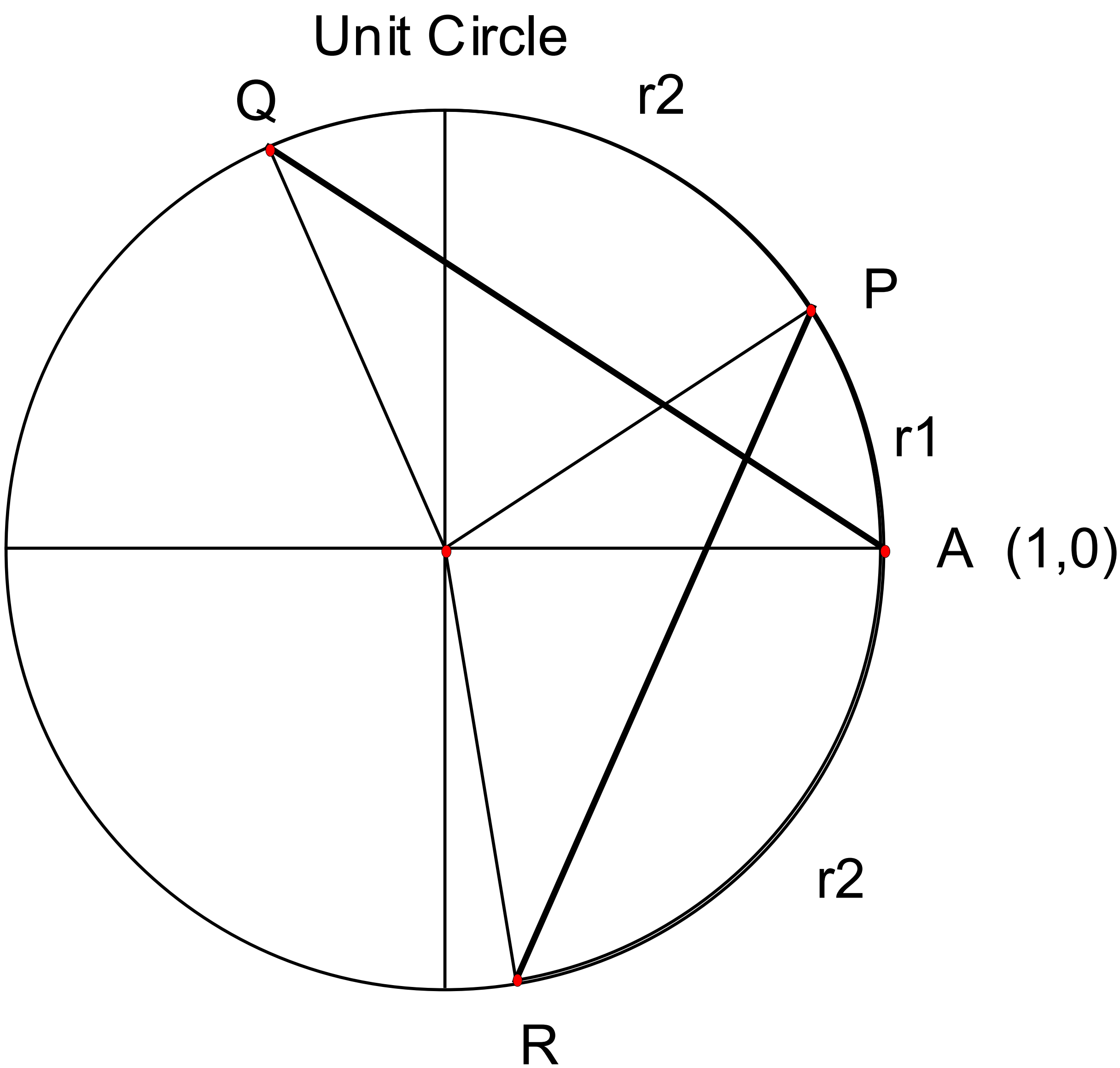
Next we add an arbitrary point Q in the 2nd quadrant and label the angular size of arc $\widehat{PR}$ r2.



Next we construct a point R in the fourth quadrant such that $\widehat{PQ} = \widehat{AR}$



Since r1+r2 = r2+r1, we have $\widehat{AQ} = \widehat{PR}$ and so also $\overline{AQ} = \overline{PR}$



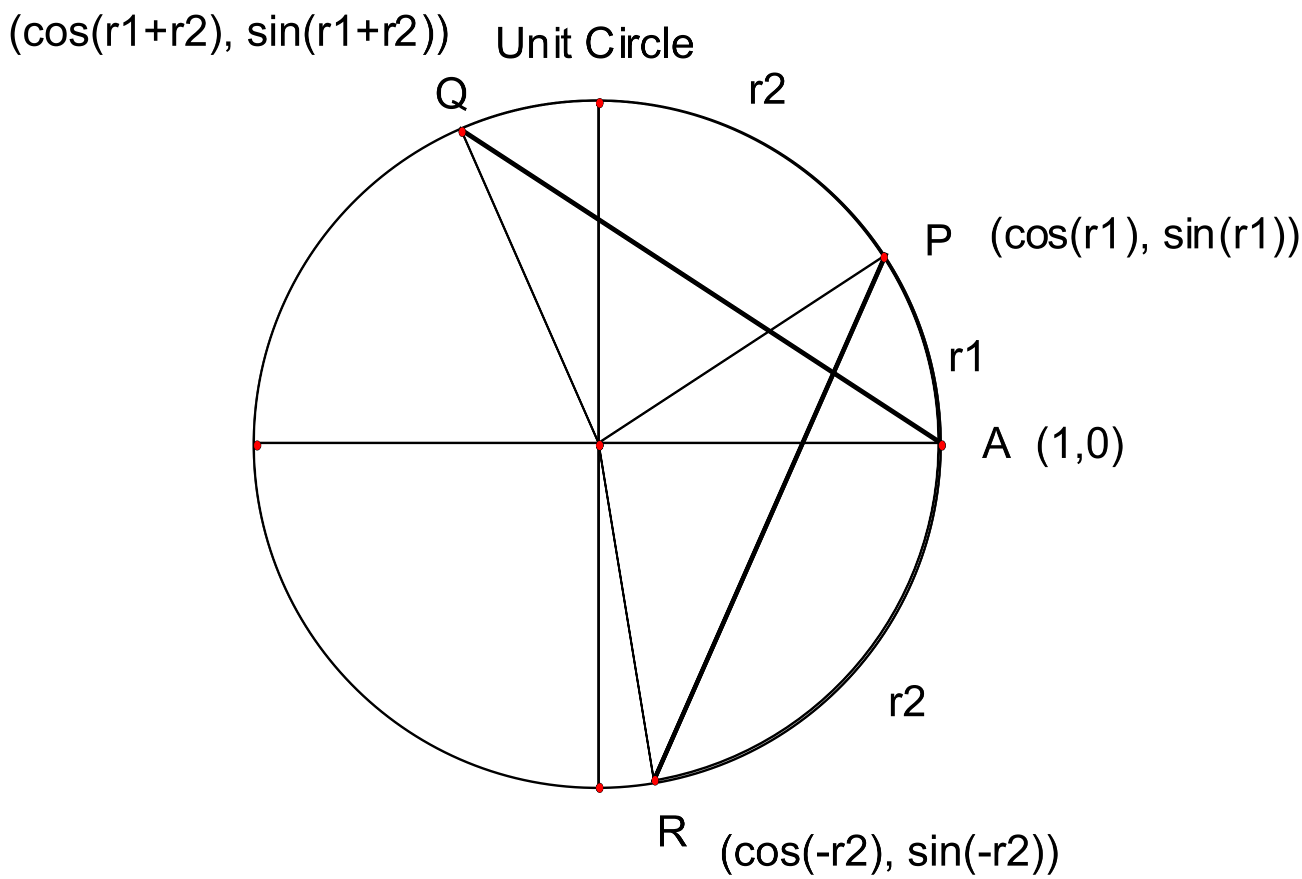
That's the geometric setup for our proof.

The angle measure of an included
The coordinates of A are (1, 0)

The coordinates of Q are (cos(r1+r2), sin(r1+r2))

The coordinates of P are (cos(r1), sin(r1))

The coordinates of R are (cos(-r2), sin(-r2))



Using the distance formula and setting $\overline{AQ} = \overline{PR}$ we have:

$$\sqrt{\left[\cos\left(r_1 + r_2\right) - 1\right]^2 + \left[\sin\left(r_1 + r_2\right) - 0\right]^2} =$$
$$\sqrt{\left[\cos\left(r_1\right) - \cos\left(-r_2\right)\right]^2 + \left[\sin\left(r_1\right) - \sin\left(-r_2\right)\right]^2}$$

Squaring both sides:

$$\left[\cos\left(r_1 + r_2\right) - 1\right]^2 + \sin^2\left(r_1 + r_2\right) =$$
$$\left[\cos\left(r_1\right) - \cos\left(r_2\right)\right]^2 + \left[\sin\left(r_1\right) + \sin\left(r_2\right)\right]^2$$

Expanding:

$$\cos^2\left(r_1 + r_2\right) - 2\cos\left(r_1 + r_2\right) + 1 + \sin^2\left(r_1 + r_2\right) =$$
$$\cos^2\left(r_1\right) - 2\cos\left(r_1\right)\cos\left(r_2\right) + \cos^2\left(r_2\right) +$$
$$\sin^2\left(r_1\right) + 2\sin\left(r_1\right)\sin\left(r_2\right) + \sin^2\left(r_2\right)$$

$$\left[\cos^2\left(r_1 + r_2\right)\right] - 2\cos\left(r_1 + r_2\right) + 1 + \left[\sin^2\left(r_1 + r_2\right)\right] =$$
$$\left[\cos^2\left(r_1\right)\right] - 2\cos\left(r_1\right)\cos\left(r_2\right) + \left[\cos^2\left(r_2\right)\right] +$$
$$\left[\sin^2\left(r_1\right)\right] + 2\sin\left(r_1\right)\sin\left(r_2\right) + \left[\sin^2\left(r_2\right)\right]$$

$$-2\cos(r_1+r_2)+2=$$

$$-2\cos(r_1)\cos(r_2)+1+2\sin(r\ )\sin(r_2)+1$$

Subtracting 2 from each side and dividing by -2

$$\cos(r_1+r_2)=\cos(r_1)\cos(r_2)-\sin(r_1)\sin(r_2)$$

This is the cosine summation formulae

$$\cos(x+y)=\cos(x)\cos(y)-\sin(x)\sin(y)$$

Plugging in -y for y and simplifying using odd/even identities

$$\cos(x-y)=\cos(x)\cos(y)+\sin(x)\sin(y)$$

We can write this

$$\cos(x\pm y)=\cos(x)\cos(y)\mp\sin(x)\sin(y)$$

The derivation for sin(x+y)

We use our co-function identity

$$\cos\left(90^\circ - (x+y)\right) = \sin(x+y)$$

We re-arrange the left side and expand

$$\sin(x+y) = \cos\left(\left(90^\circ - x\right) - y\right) =$$

$$\cos\left(90^\circ - x\right)\cos(-y) - \sin\left(90^\circ - x\right)\sin(-y) =$$

$$\sin(x)\cos(y) - \cos(x)(-1)\sin(y) =$$

$$\sin(x)\cos(y) + \cos(x)\sin(y)$$

$$\sin(x+y) = \sin(x)\cos(y) + \sin(y)\cos(x)$$

Substituting -y for y

$$\sin\left(x + {}^{-}y\right) = \sin(x)\cos(-y) + \sin(-y)\cos(x)$$

Using odd/even identities it becomes

$$\sin(x-y) = \sin(x)\cos(y) - \sin(y)\cos(x)$$

To find tan(x+y):

$$\tan(x+y)=\frac{\sin(x+y)}{\cos(x+y)}=\frac{\sin x\cos y+\sin y\cos x}{\cos x\cos y-\sin x\sin y}$$

$$\frac{\sin x\cos y+\sin y\cos x}{\cos x\cos y-\sin x\sin y}\cdot\frac{\dfrac{1}{\cos x\cos y}}{\dfrac{1}{\cos x\cos y}}=\frac{\dfrac{\sin x}{\cos x}+\dfrac{\sin y}{\cos y}}{1-\dfrac{\sin x\sin y}{\cos x\cos y}}=\frac{\tan x+\tan y}{1-\tan x\tan y}$$

so:

$$\tan(x+y)=\frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}$$

Similarly:

$$\tan(x-y)=\frac{\tan(x)-\tan(y)}{1+\tan(x)\tan(y)}$$

Using the formulas

(a) $\cos 75^\circ = \cos\left(45^\circ + 30^\circ\right)$

(b) $\cos\left(\frac{\pi}{12}\right) = \cos\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$

Double Angle Formulas:

If

$$\sin(x+y)=\sin(x)\cos(y)+\sin(y)\cos(x)$$

then

$$\sin(x+x)=\sin(x)\cos(x)+\sin(x)\cos(x)$$

or simplified

$$\sin(2x)=2\sin(x)\cos(x)$$

Similarly

$$\cos(2x)=\cos^2(x)-\sin^2(x)$$

Using the Pythagorean identity we get two other useful forms:

$$\cos(2x)=2\cos^2(x)-1$$

and

$$\cos(2x)=1-2\sin^2(x)$$

The tangent double angle then becomes

$$\tan(2x)=\frac{2\tan(x)}{1-\tan^2(x)}$$

HW
7.2: 5-8, 32, 35, 43
7.3: 5-8, 17, 21, 25