The Dot Product of Two Vectors

We've found that like numbers vectors can be added, subtracted and multiplied by another number.

The question arises as to whether there is any way to combine two vectors in a product. There are two ways to do this called the Dot Product and the Cross Product.
These are written as follows: (see next page).

We will not be covering the cross product in this class, but anyone taking Physics will come across it, so we will take a quick look at the cross product near the end of class if there is time.

$\vec{V} \cdot \vec{U}$ Dot-Product

$\vec{V} \times \vec{U}$ Cross-Product

We start by defining the dot product of two vectors in component form

$$\vec{v} = \langle x_1, y_1 \rangle$$
$$\vec{u} = \langle x_1, y_1 \rangle$$
$$\vec{v} \cdot \vec{u} = x_1 x_2 + y_1 y_2$$

Note that the pattern here persists into 3 dimensions

$$\vec{V} \cdot \vec{U} = V_x U_x + V_y U_y + V_z U_z$$

Note that the result of the dot-product is not a vector, it is a number.

In order to show a rather remarkable property of the dot product we develop a few properties of the dot product.

Note that because multiplication is commutative, so is the dot product
$$\vec{u} \cdot \vec{v} = x_2 x_1 + y_2 y_1 = x_1 x_2 + y_1 y_2 = \vec{v} \cdot \vec{u}$$

Similarly we can show that the dot product distributes over vector addition.
$$\left(\vec{u} + \vec{v}\right) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$$

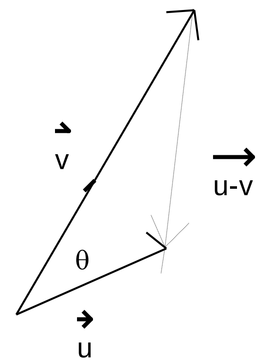
Multiplication by a scalar is associative

$$\left(c\vec{v}\right) \cdot \vec{u} = c\left(\vec{v} \cdot \vec{u}\right) = \vec{v} \cdot \left(c\vec{u}\right)$$

Finally we note that

$$\vec{u} \cdot \vec{u} = x_1 x_1 + y_1 y_1 = x_1^2 + y_1^2 = \left(\sqrt{x_1^2 + y_1^2}\right)^2 = \left|\vec{u}\right|^2$$

From this diagram



We apply the law of cosines

$$\left|\vec{u}-\vec{v}\right|^2 = \left|\vec{u}\right|^2 + \left|\vec{v}\right|^2 - 2\left|\vec{u}\right|\left|\vec{v}\right|\cos\theta$$

The left and side distributes giving

$$\left|\vec{u}-\vec{v}\right|^2 = \left(\vec{u}-\vec{v}\right)\cdot\left(\vec{u}-\vec{v}\right) = \vec{u}\cdot\vec{u}-\vec{u}\cdot\vec{v}-\vec{v}\cdot\vec{u}+\vec{v}\cdot\vec{v} = \left|\vec{u}\right|^2 - 2\left(\vec{u}\cdot\vec{v}\right)+\left|\vec{v}\right|^2$$

Equating this with the right side we get

$$\left|\vec{u}\right|^2 - 2\left(\vec{u}\cdot\vec{v}\right)+\left|\vec{v}\right|^2 = \left|\vec{u}\right|^2 + \left|\vec{v}\right|^2 - 2\left|\vec{u}\right|\left|\vec{v}\right|\cos\theta$$
$$-2\left(\vec{u}\cdot\vec{v}\right) = -2\left|\vec{u}\right|\left|\vec{v}\right|\cos\theta$$
$$\vec{u}\cdot\vec{v} = \left|\vec{u}\right|\left|\vec{v}\right|\cos\theta$$

This gives us an alternate way of calculating the dot product if we know the norms of the vectors and the angle between them. It also gives us a way to find the angle between them if we just know the component form by solving for θ

$$\theta = \cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{\left|\vec{u}\right|\left|\vec{v}\right|}\right)$$

Examples:

$\vec{u} = \langle 3,5 \rangle$ and $\vec{v} = \langle 2,-8 \rangle$

What is the angle between the two vectors?

$$\theta = \cos^{-1}\left(\frac{3\cdot 2+5\cdot -8}{\sqrt{34}\cdot\sqrt{68}}\right) = \cos^{-1}\left(\frac{-34}{\sqrt{34\cdot 68}}\right) = \cos^{-1}\left(\frac{-1}{\sqrt{2}}\right) = 135^\circ$$

$\vec{u} = \langle 2,1 \rangle$ and $\vec{v} = \langle -1,2 \rangle$

What is the angle between the two vectors?

Since $2\cdot -1+1\cdot 2=0$ $\qquad \cos^{-1}(0) = 90^\circ$

So the vectors are perpendicular!

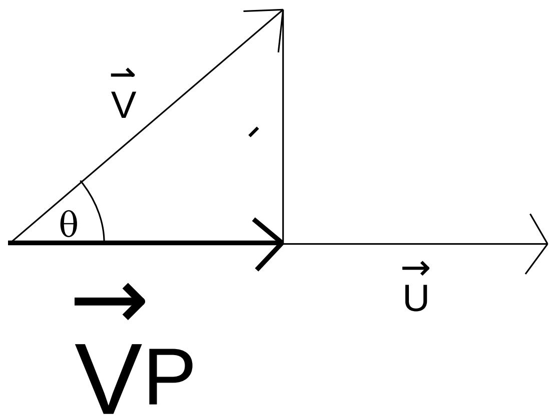
Note that whenever two vectors are perpendicular [**orthogonal]**

their dot product will be zero.

Projections

We project one vector onto another as shown in the picture below

Here Vector V is projected onto Vector U



Note that:

$$\left|\overrightarrow{V_P}\right| = \left|\vec{V}\right|\cos(\theta)$$

Also $\overrightarrow{V_P}$ has the same direction as $\vec{U}$

We can make $\vec{U}$ into a Unit Vector, a vector with magnitude 1 as follows:

$$\vec{U}_{Unit} = \frac{\vec{U}}{\left|\vec{U}\right|}$$

This gives us a definition of a Projection of P onto U:

$$\vec{V}_{P\to\vec{U}} = \left|\vec{V}\right|\cos(\theta)\frac{\vec{U}}{\left|\vec{U}\right|} = \frac{\left|\vec{V}\right|}{\left|\vec{U}\right|}\cos(\theta)\vec{U}$$

In terms of the dot product, this becomes

$$\vec{V}_{P\to\vec{U}} = \left(\frac{\vec{V}\cdot\vec{U}}{\left|\vec{U}\right|^2}\right)\vec{U}$$

Example

Find the components of u along v

$$\vec{u} = \langle 1,4\rangle \text{ and } \vec{v} = \langle -2,1\rangle$$

What are the components of the
projection of u onto v?

$$\vec{u}_{P\to\vec{v}} = \left(\frac{\vec{u}\cdot\vec{v}}{\left|\vec{u}\right|^2}\right)\vec{v} = \frac{1\cdot -2+4\cdot 1}{\left(\sqrt{4+1}\right)^2}\langle -2,1\rangle = \frac{2}{5}\langle -2,1\rangle = \left\langle \frac{-4}{5}, \frac{2}{5}\right\rangle$$

If instead of $\overrightarrow{U}$ we use $\vec{i}$ or $\vec{j}$

Projecting onto the *x* and *y* axis gives us

$$\overrightarrow{V}_{P\to\vec{i}} = \left(\overrightarrow{V}\cdot\vec{i}\right)\vec{i}$$

$$\overrightarrow{V}_{P\to\vec{j}} = \left(\overrightarrow{V}\cdot\vec{j}\right)\vec{j}$$

That is we can find the components of $\overrightarrow{V}$ using the dot product with the formula

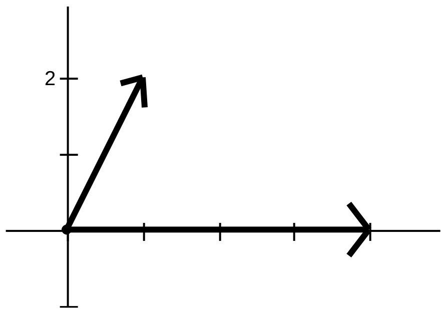
$$\overrightarrow{V} = \left\langle \overrightarrow{V}\cdot i, \overrightarrow{V}\cdot j \right\rangle$$

Basis and Vector Space

We have this important relationship that any plane vector can be described by this equation

$\vec{v} = a\vec{i} + b\vec{j}$ where the component form of the vector is $\langle a, b \rangle$.

This is not the only way we can break down a vector.
For example, if we have the two vectors that are not parallel



$\vec{h} = \langle 4, 0 \rangle$

$\vec{k} = \langle 1, 2 \rangle$

We can decompose any other vector as a linear combination of these vectors.

Example:
$\vec{v} = \langle 11, 6 \rangle = 2\vec{h} + 3\vec{k}$

Any set of vectors, eg $\{\vec{i}, \vec{j}\}$ or $\{\vec{h}, \vec{k}\}$ is called a **basis.**

The set of all vectors that can be created from any linear combination of a basis,

$\vec{v} = a\vec{i} + b\vec{j}$

is called a **vector space**.

Because the basis vectors in $\{\vec{i}, \vec{j}\}$ are orthogonal to each other, the dot product can be used to decompose a vector into a linear combination of them.

Note that for a plane two vectors are required, but also sufficient.
We say the **dimension** of the space is 2.

Most of this generalizes to 3 or more dimensions.

Example:

What is the angle between the three dimensional vectors?

$\langle 3,2,-1\rangle$ and $\langle -1,0,4\rangle$

$$\theta=\cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{\left|\vec{u}\right|\left|\vec{v}\right|}\right)=\cos^{-1}\left(\frac{-3+0-4}{\sqrt{3^2+2^2+(-1)^2}\cdot\sqrt{(-1)^2+0^2+4^2}}\right)=$$

$$\cos^{-1}\left(\frac{-7}{\sqrt{14}\cdot\sqrt{17}}\right)=\cos^{-1}\left(\frac{-7}{\sqrt{14}\cdot\sqrt{17}}\right)\approx\cos^{-1}(-.45373)\approx 117^\circ$$

To convert the components of a vector from one basis to another requires a linear transformation. A linear transformation in 2 dimensions is represented by a 2x2 matrix:

Example:

$$\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$$

The study of linear transformations is the main subject of **linear algebra**, which you are ready for, but is usually taught after calculus.