Section 10-5 Inverses of Matrices

Inverse of a Matrix

If the product of two matrices is the identity matrix I then the two matrices are inverses.

Example:

$$A = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$$

$$B = \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}$$

$$AB = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix} = \begin{pmatrix} 2\cdot 3 + 1\cdot {}^{-}5 & 2\cdot {}^{-}1 + 1\cdot {}^{-}2 \\ 5\cdot 3 + 3\cdot {}^{-}5 & {}^{-}1\cdot 5 + 2\cdot 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

For a 2x2 matrix there is a simple formula for the inverse

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

This is easy to verify.

$$AA^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{ad - bc}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} =$$

$$\frac{1}{ad - bc}\begin{pmatrix} ad + {}^{-}cd & -ab + ab \\ cd + -dc & -cb + ad \end{pmatrix} = \frac{1}{ad - bc}\begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

To find the inverse of any matrix we can do the following:

Convert using row reduction rules:

$$\begin{pmatrix} a & b & 1 & 0 \\ c & d & 0 & 1 \end{pmatrix} -> \begin{pmatrix} 1 & 0 & a' & b' \\ 0 & 1 & c' & d' \end{pmatrix}$$

Note that if $ac - bd = 0$ then the system has no solution.

Uses for the Inverse:

If we have a system of linear equations such as

$$2x+3y=-1$$
$$x-4y=5$$

We can write this as a matrix equation.

$$\begin{pmatrix} 2 & 3 \\ 1 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} -1 \\ 5 \end{pmatrix}$$

If we multiply the left and right sides of the equation by the inverse of the matrix, then we will get

$$A^{-1}\begin{pmatrix} 2 & 3 \\ 1 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} x \\ y \end{pmatrix}=A^{-1}\begin{pmatrix} -1 \\ 5 \end{pmatrix}$$

So by finding the inverse of the matrix we can solve the system of equations:

$$A^{-1}=\frac{1}{2\cdot{}^{-}4-1\cdot 3}\begin{pmatrix} -4 & -3 \\ -1 & 2 \end{pmatrix}=\frac{1}{{}^{-}11}\begin{pmatrix} -4 & -3 \\ -1 & 2 \end{pmatrix}=\begin{pmatrix} 4/11 & 3/11 \\ 1/11 & -2/11 \end{pmatrix}$$

$$A^{-1}\begin{pmatrix} -1 \\ 5 \end{pmatrix}=\begin{pmatrix} 4/11 & 3/11 \\ 1/11 & -2/11 \end{pmatrix}\begin{pmatrix} -1 \\ 5 \end{pmatrix}=\begin{pmatrix} -4/11+15/11 \\ -1/11-10/11 \end{pmatrix}=\begin{pmatrix} 11/11 \\ -11/11 \end{pmatrix}=\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$