Connections

Complex Numbers
Polynomials
Exponential & Log Functions
Hyperbolic Functions
Trigonometric Functions

Connection between Exponential functions and polynomials

$$e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!} = \frac{1}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

Note that

$$e^0 = 1 + 0 + 0 + \cdots$$

$$e^1 = \sum_{i=0}^{\infty} \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots =$$

$$1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \cdots \approx 2.71818$$

Hyperbolic Functions

Hyperbolic Sine

$$\sinh(x)=\frac{e^x-e^{-x}}{2}$$

Hyperbolic Cosine

$$\cosh(x)=\frac{e^x+e^{-x}}{2}$$

$$e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots$$

$$e^{-x}=1-\frac{x}{1!}+\frac{x^2}{2!}-\frac{x^3}{3!}+\frac{x^4}{4!}-\cdots$$

$$\sinh(x)=\frac{x}{1!}+\frac{x^3}{3!}+\frac{x^5}{5!}+\frac{x^7}{7!}+\cdots$$

$$\cosh(x)=1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\cdots$$

Trigonometric Functions

$$\sin(x)=\frac{x}{1!}-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots$$

$$\cos(x)=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots$$

Thinking about complex numbers

$$e^{y+ix} = e^y e^{ix}$$
$$e^{ix} = 1 + \frac{ix}{1!} + \frac{i^2 x^2}{2!} + \frac{i^3 x^3}{3!} + \frac{i^4 x^4}{4!} + \cdots$$
$$e^{ix} = 1 + \frac{ix}{1!} - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \cdots$$
$$e^{ix} = \frac{ix}{1!} - \frac{ix^3}{3!} + \frac{ix^5}{5!} - \frac{ix^7}{7!} \cdots + 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} \cdots$$
$$e^{ix} = i\left(\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}\right) \cdots + 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} \cdots$$
$$e^{ix} = \cos x + i \sin x$$

This is called Euler's Formula

When we get to trigonometry we will be using a scale other than degrees called radians. This formula works for radians.

$$180^\circ = \pi \ radians$$

$$\cos(\pi) = -1$$
$$\sin(\pi) = 0$$

So we have Euler's identity

$$e^{i\pi} = -1$$
$$or$$
$$e^{i\pi} + 1 = 0$$

Finally, using these equations we find that

$$\sin x = \frac{e^{ix} - e^{-ix}}{2}$$
$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

Connections between Hyperbolic and Trigonometric functions

Identities

$$(\cos x)^2 + (\sin x)^2 = 1$$
$$(\cosh x)^2 - (\sinh x)^2 = 1$$

Note that

$$x^2 + y^2 = 1$$

is the equation of a circle

$$x^2 - y^2 = 1$$

is the equation of a hyperbola

Which might help to explain why Trigonometric functions are sometimes called circular functions.