Lesson Plan 3 - Indefinite Integrals 5.3

1) Take attendance, any new students?

2) Quiz a week from today on definite and indefinite integrals

3) Questions on the homework or the work sheet?

Recall that if $F'(x) = f(x)$ then $\int_a^b f(x)dx = F(b) - F(a) = \left[F(x)\right]_a^b$

It is handy to have some notation to refer to the indefinite integral, so we have

$$F(x) = \int f(x)\,dx$$

This is called an indefinite integral. Unlike the definite integral, this is not a number. This is a function of x or rather a family of functions.

For example:

$\int x\,dx = \frac{x^2}{2} + C$ where C is any constant $C \notin \mathbb{R}$

The connection between an indefinite integral and a definite integral is

$$\int_a^b f(x)dx = \left[\int f(x)dx\right]_a^b$$

Some examples of evaluating indefinite Integrals

1)

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$ for $x \neq -1$

As a check $\frac{d}{dx}\frac{x^{n+1}}{n+1} + C = x^n + 0 = x^n$

2) $\int e^x dx = e^x + C$

As a check $\frac{d}{dx}e^x + C = e^x + 0 = e^x$

3) $\int \sin(x)dx = -\cos(x) + C$

As a check $\frac{d}{dx} - \cos(x) + C = -\frac{d}{dx}\cos(x) + \frac{d}{dx}C = -\sin(x) + 0 = -\sin(x)$

4) $\int \frac{1}{x}dx = \ln|x| + C$

As a check $\frac{d}{dx}\ln|x| + C = \frac{1}{|x|} + 0 = \frac{1}{|x|}$

Note the following important pattern:

$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + C$ This comes up a lot, eg.

$$\int \frac{2x+1}{x^2+x+4}dx$$

Since $\frac{d}{dx}\left(x^2+x+4\right) = 2x+1$

$$\int \frac{2x+1}{x^2+x+4}dx = \ln\left|x^2+x+4\right| + C$$

Another pattern

Since $\frac{d}{dx}f(x)^{-n}=(-n)f(x)^{-n+1}f'(x)=-n\frac{f'(x)}{f(x)^{n-1}}$

we have $\int\frac{f'(x)}{f(x)^{n}}dx=-\left(\frac{1}{n-1}\right)\frac{1}{f(x)^{n-1}}+C$

Example:

$$\int\frac{x}{\left(x^2+1\right)^2}dx$$

We know that $\frac{d}{dx}\left(x^2+1\right)=2x$

So we have $\int\frac{x}{\left(x^2+1\right)^2}dx=\frac{1}{2}\int\frac{2x}{\left(x^2+1\right)^2}dx=\frac{1}{2}\left(-\frac{1}{1}\right)\frac{1}{x^2+1}+C=\left(-\frac{1}{2}\right)\frac{1}{x^2+1}$

Example, find: $\int\sec(x)\tan(x)dx$

$$\int\sec(x)\tan(x)dx=\int\frac{1}{\cos(x)}\cdot\frac{\sin(x)}{\cos(x)}dx=\int\frac{\sin(x)}{\cos(x)^2}dx=-\int\frac{-\sin(x)}{\cos(x)^2}dx$$

But $\left[\cos(x)\right]'=-\sin(x)$

So we have $\int\sec(x)\tan(x)dx=-\left(-\frac{1}{\cos(x)}\right)+C=\frac{1}{\cos(x)}+C=\sec(x)+C$

Yet Another Pattern

Recall that $\frac{d}{dx}\sin^{-1}(x)=\frac{1}{\sqrt{1-x^2}}$

How do we know this?

$$\frac{dy}{dx}=1\Big/\frac{dx}{dy}$$

If $x=\sin(y)$ then $\frac{dx}{dy}=\cos(y)$

$$\frac{dy}{dx}=\frac{1}{\cos(y)}=\frac{1}{\sqrt{1-\sin^2(y)}}=\frac{1}{\sqrt{1-x^2}}$$

So clearly we have $\int\frac{1}{\sqrt{1-x^2}}\,dx=\sin^{-1}(x)+C$

But likewise $\int\frac{f'(x)}{\sqrt{1-f(x)^2}}\,dx=\sin^{-1}(f(x))+C$

Example: $\int\frac{x}{\sqrt{1-x^4}}\,dx$

Since $\frac{d}{dx}x^2=2x$

$$\int\frac{x}{\sqrt{1-x^4}}\,dx=\frac{1}{2}\int\frac{2x}{\sqrt{1-(x^2)^2}}\,dx=\frac{1}{2}\sin^{-1}(x^2)+C$$

A similarly useful formula

$$\int\frac{f'(x)}{1+f(x)^2}\,dx=\tan^{-1}(x)+C$$

Simplifying before integrating, a problem from the book, Example 6 p. 360.

$$\int_1^9 \frac{2t^2+t^2\sqrt{t}-1}{t^2}dx = \int_1^9 2+t^{1/2}-\frac{1}{t^2}\,dx = \left[2t+\frac{2}{3}t^{3/2}+\frac{1}{t}\right]_1^9 =$$

$$18+\frac{2}{3}\cdot 27+\frac{1}{9}-\left(2+\frac{2}{3}+1\right)=36-3+\frac{1}{9}-\frac{2}{3}=33+\frac{1-6}{9}=33-\frac{5}{9}=32\frac{4}{9}$$

Give Students Handout 3 with some examples to try:

Net Change

We can rewrite

$$\int_a^b f(x)dx = F(b) - F(a)$$

as

$$\int_a^b F'(x)dx = F(b) - F(a)$$

Where $F'(x)$ is the rate of change of $F(x)$

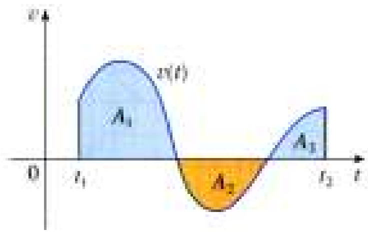
Example:

Consider a reservoir which has water flowing into or out of it at a rate of $V'(t)$

So $V(t_2) - V(t_1) = \int_{t_1}^{t_2} V'(t)dt$ is the change in the amount of water between time t_1 and t_2.

Displacement vs. distance traveled.

Consider a train that travels back and forth on a straight rail at a velocity according to this graph.



If you wish to know it's displacement then

$$\text{displacement} = A_1 - A_2 + A_3 = \int_{t_1}^{t_2} v(t)\,dt$$

If instead you want to know the distance traveled then you want

$$\text{distance} = A_1 + A_2 + A_3 = \int_{t_1}^{t_2} |v(t)|\,dt$$

Example from the book:

A particle has velocity $v(t) = t^2 - t - 6$
Find the displacement and distance traveled between 1 and 4 seconds:

displacement = $\int_1^4 (t^2 - t - 6)\,dt =$

$$\left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_1^4 = \frac{64}{3} - 8 - 24 - \left(\frac{1}{3} - \frac{1}{2} - 6\right) = -\frac{9}{2}$$

distance = $\int_1^4 \left|t^2 - t - 6\right| dt$

Find where the direction changes: $t^2 - t - 6 = (t-3)(t+2) = 0$
So it changes at -2 and 3 seconds.

$$\int_1^4 \left|t^2 - t - 6\right| dt = \int_1^3 \left|t^2 - t - 6\right| dt + \int_3^4 \left|t^2 - t - 6\right| dt =$$

$$\left|\left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_1^3\right| + \left|\left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_3^4\right| =$$

$$\left|9 - \frac{9}{2} - 18 - \left(\frac{1}{3} - \frac{1}{2} - 6\right)\right| + \left|\frac{64}{3} - 8 - 24 - \left(9 - \frac{9}{2} - 18\right)\right| =$$

$$\left|-\frac{22}{3}\right| + \left|\frac{17}{6}\right| = \frac{44}{6} + \frac{17}{6} = \frac{61}{6}$$