Lesson Plan 21 - Limits

1) Take attendance
2) Return Homework
3) Homework questions?
4) Administer Quiz

Definition of a Limit

$\lim_{x \to c} f(x) = L$ iff for each ε>0 there exists δ>0 such that
if $0 < |x - c| < \delta$ then $|f(x) - L| < \varepsilon$

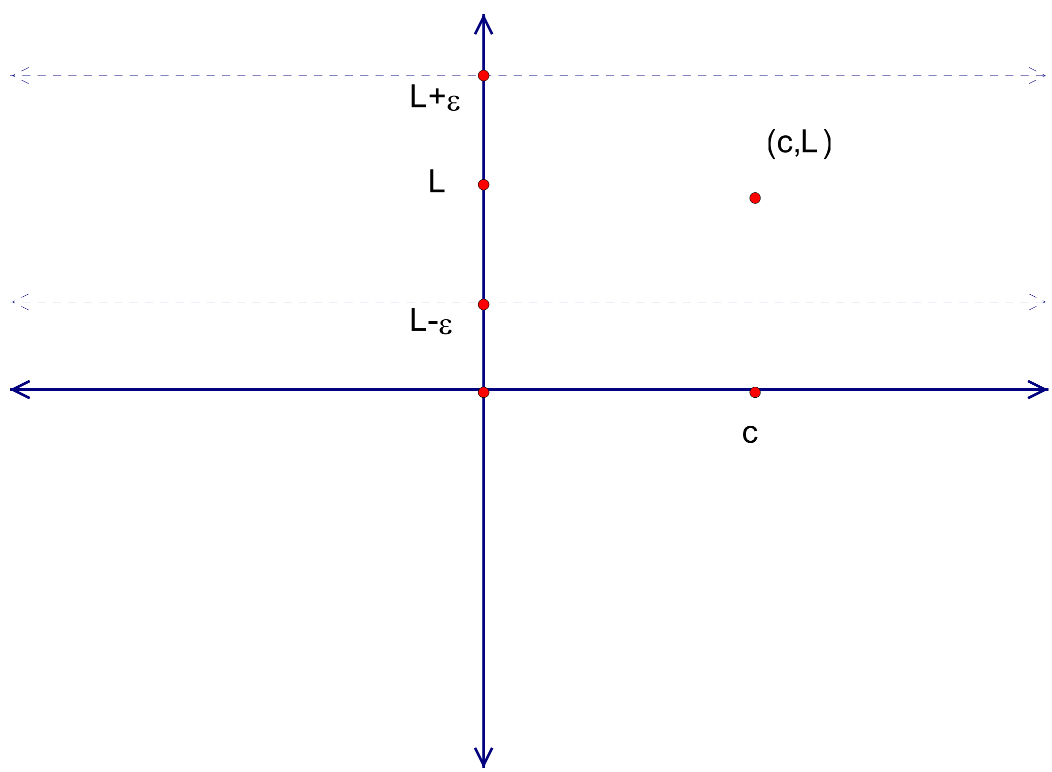
What does this mean?

Note this doesn't tell us how to find a limit. We have to find the limit ourselves.
But this gives us a way to prove that the number we think is a limit, is one.

First we choose some small ε, which brackets our limit L with an interval on *y*:

$(L - \varepsilon, L + \varepsilon)$

Like this:



We are saying here we want to limit our function to values on this interval $(L-\varepsilon, L+\varepsilon)$.

We do this by finding a δ such that on the interval $(c-\delta, c+\delta)$ $f(x) \in (L-\varepsilon, L+\varepsilon)$

A) $0 < |x-c| < \delta$ chooses the δ

B) $|f(x)-L| < \varepsilon$ means that $f(x) \in (L-\varepsilon, L+\varepsilon)$

So if we can show that A implies B, we have shown that L is the limit.

Let's take a step back and think about why this is!

We do not specify what ε is.
ε can be arbitrarily small as long as it isn't zero.
That might seem odd.
A limit really has nothing to do with a function's value at c.
A function often isn't even defined at c.
It has to do with what value the function can get arbitrarily close to, no matter how small.

By showing that there always is such a δ, we've shown that we can make $f(x)$ get arbitrarily close to L.

Let's see what a so called ε, δ proof looks like.

Example:

Prove that $\lim_{x\to 2}(2x-1) = 3$

Let ε>0. We must show there exists δ>0 such that

$$0 < |x-2| < \delta \rightarrow |(2x-1)-3| = |2x-4| < \varepsilon$$

Since $|2x-4| = 2|x-2|$ we choose $\delta = \frac{\varepsilon}{2}$

Then we have $0 < |x-2| < \frac{\varepsilon}{2} \rightarrow |2x-4| < \varepsilon$
Thus we have shown that 3 is the limit.

Let's try one slightly more complicated.

Example:

Prove that $\lim_{x\to 4}\sqrt{x} = 2$

Since we can choose δ arbitrarily, we choose δ to be any number δ<4 and $\delta < 2\varepsilon$

Since δ<4 and $0 < |x-4| < \delta$ it should be clear that $x > 0$, and so $\sqrt{x}$ is real.
From here we have $x-4 = \left(\sqrt{x}+2\right)\left(\sqrt{x}-2\right) < \delta$.

But $2 < \left|\sqrt{x}+2\right|$ so $2\left|\sqrt{x}-2\right| < \delta$ or $\left|\sqrt{x}-2\right| < \dfrac{\delta}{2}$.

But since we've made $\dfrac{\delta}{2} < \varepsilon$ we have $\left|\sqrt{x}-2\right| < \varepsilon$ which is what we needed to show.

That's all.