# Lesson Plan 18 - Supplementary Differential Equations

We classify Differential equations in a number of ways.

Equations where the function is of just one variable are called "ordinary".
Equations where the function is of two or more variables are called "Partial differential Equations".

We will only deal with ordinary differential equations in this class.

Differential equations are classified by the highest derivative.
So for example if a second derivative of the function is the highest derivative, the equation is a 2nd degree equation.

A differential equation where the function and its derivatives are each only multiplied by a constant are linear. Any linear combination of two solutions to a linear differential equation is a solution.

If the function or a derivative of the function is multiplied by a function of x, then the equation is non-linear.

We will examine here how to solve the first degree non-linear equation
$y' + P(x)y = Q(x)$

We start with a simpler equation
$y' + ay = 0$

Since $e^{ax} \neq 0$ for any $x$ we can multiply both sides of the equation by it, giving:

$e^{ax}y' + ae^{ax}y = 0$

Note that using the product rule in reverse we have

$\frac{d}{dx}\left[e^{ax}y\right]$
so $e^{ax}y = C$

giving us the solution $y = Ce^{-ax}$.

Given an initial condition that $y(x_0) = y_0$ we find that
$y_0 = Ce^{-ax_0}$ or $C = y_0e^{ax_0}$ with the final solution being

$y(x) = y_0e^{a(x_0 - x)}$

Example: $y'+3y=0$ with $y(1)=\pi$

$y(x)=\pi e^{3(1-x)}$

Now look at $y'+ay=b$

$e^{ax}y'+ae^{ax}y=be^{ax}$

Note that using the product rule in reverse we have

$$\frac{d}{dx}\left[e^{ax}y\right]=be^{ax}$$

so $e^{ax}y=\frac{b}{a}e^{ax}+C$

giving us the solution $y(x)=\frac{b}{a}+Ce^{-ax}$.

Given an initial condition that $y(x_0)=y_0$ we find that

$y_0=\frac{b}{a}+Ce^{-ax_0}$ or $C=\left(y_0-\frac{b}{a}\right)e^{ax_0}$ with the final solution being

$$y(x)=\frac{b}{a}+\left(y_0-\frac{b}{a}\right)e^{a(x_0-x)}$$

Example: $y'(x)-\frac{1}{3}y(x)=1$ with $y\left(\frac{1}{2}\right)=-2$

$y(x)=-3+e^{(x-1/2)/3}$

Finally we look $y' + P(x)y = Q(x)$

We define $H(x) = \int_{x_0}^{x} P(t)\,dt \to H'(x) = P(x)$

Multiplying through by $e^{H(x)}$ we get

$e^{H(x)}y' + e^{H(x)}P(x)y = e^{H(x)}Q(x)$ since $H'(x) = P(x)$

when we gain using the product rule in reverse we get:

$$\frac{d}{dx}\left[e^{H(x)}y(x)\right] = e^{H(x)}Q(x)$$

Integrating on both sides

$$e^{H(x)}y(x) = \int_{x_0}^{x} Q(s)e^{H(s)}ds + C$$

so $y(x) = e^{-H(x)}\left[\int_{x_0}^{x} Q(x)e^{H(s)}ds + C\right]$

For initial conditions $y(x_0) = y_0$

Since $e^{-H(x_0)} = e^0 = 1$ and $\int_{x_0}^{x_0} Q(s)e^{H(s)}ds = 0$

$C = y_0$ so the final solution is

$$y(x) = e^{-H(x)}\left[\int_{x_0}^{x} Q(x)e^{H(s)}ds + y_0\right]$$

Example:

$y'(x)+2y(x)=x^2$ where $y(0)=1$

$P(x)=2$ and $Q(x)=x^2$ so

$H(x)=\int_0^x 2\,ds=2x$ and $y(x)=e^{-2x}\left(1+\int_0^x s^2e^{2s}ds\right)$

Evaluating the integral by twice integrating by parts we have

$\int_0^x s^2e^{2s}ds=e^{2x}\left(\frac{1}{2}x^2-\frac{1}{2}x+\frac{1}{4}\right)-\frac{1}{4}$

So $y(x)=e^{-2x}+\frac{1}{2}x^2-\frac{1}{2}x+\frac{1}{4}$