Lesson Plan 17 - Quiz & Differential Equations 7.1

1) Take attendance
2) Homework questions?
3) Administer Quiz

Differential Equations

What is a Differential equation.

It is an equation that includes an unknown function and one more of its derivatives.

An example might be:

$$f(x) = Af'(x)$$

Because the function e^x has the unusual property that $\frac{d}{dx}e^x = e^x$ it plays an important role in solving differential equations.

Often the solution of a differential equation will be a family of solutions.
If the problem to be solved has some initial conditions, this may narrow the solution down to a single function.

We start by looking at a specific example using the growth of a population. This could be a population of people, animals or bacterial. Typically the rate of increase in a population is related to the size of the population. We write this as follows:

$$\frac{dP}{dt} = kP$$

This means that the rate of change of a population is a multiple of the current population.

As we pointed out, a likely contender is the function $P(t) = Ce^{kt}$.

We check this by noting that

$P'(t) = kCe^{kt} = kP(t)$ indicating that this is a solution. Note that 'C' is an unknown constant.

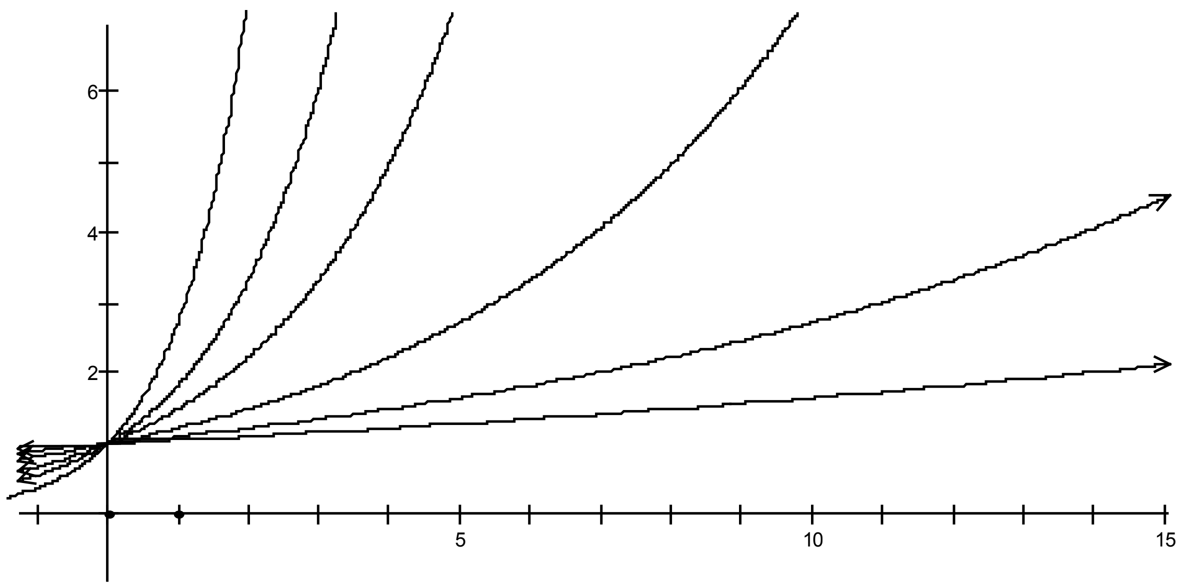
Note that if $C \le 0$ we are talking about a zero population, which is not very interesting, or a negative population which has no meaning at all, so assume that $C > 0$.

Consider, what is the population at time $t = 0$?

$P(0) = Ce^0 = C$ so C represents the initial value of the population, so if we have this initial condition, we know the value of C.

Then to set the value of k we would need one more data point.

Here are some examples for k with different values:



This differential equation will only apply for a population with an unlimited supply of resources that it needs to grow.

The maximum population that a limited but constant supply of resources can sustain is called the CARRYING CAPACITY.

As a population nears it's carrying capacity the differential equation can be amended to:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

This is called the LOGISTIC DIFFERENTIAL EQUATION.

Note that for P <<M the solution should be the same as our original equation, but for $P = M$ $\frac{dP}{dt} = 0$.

The solution to the logistic equation is a function that increases exponentially until it nears the carrying capacity at which point it levels off and becomes a constant function.

Motion of a Spring

As mentioned on Thursday, many springs will obey Hook's law that $F = -kx$ meaning the force returning the spring to its initial position is directly proportional to and in the opposite direction of its distance from its natural length.

Using the physics equation $F = ma$ we have $ma = -kx$ or $a = -\frac{k}{m}x$.

Since acceleration is the 2nd derivative of position with respect to time, this leaves us with the differential equation:

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

If you think for a minute about a function whose 2nd derivative is proportional to but the negative of the original function, you might come up with the sine and/or cosine.

With this idea in mind we try the function

$$x(t) = A\sin(rt) + B\cos(rt) + C$$

Note that here we mean by x the function $x(t)$

$$x'(t) = rA\cos(rt) - rB\sin(rt)$$

$$x''(t) = -r^2 A\sin(rt) - r^2 B\cos(rt) = -r^2\left[A\sin(rt) + B\cos(rt)\right] = -r^2 x$$

So if we set $r = \sqrt{\frac{k}{m}}$ we have the solution $x(t) = A\sin\left(\sqrt{\frac{k}{m}}t\right) + B\cos\left(\sqrt{\frac{k}{m}}t\right) + C$

If we have the initial condition that we stretch the spring a distance *D* and release it at *t*=0, making this an INITIAL VALUE PROBLEM then we have

$$x(0) = B + C = D$$

$$x'(0) = \sqrt{\frac{k}{m}}A = 0 \to A = 0$$

and we know that the acceleration at *t*=0 must be $-\frac{k}{m}D$ so

$$x''(0) = -\frac{k}{m}D = -\frac{k}{m}B \to B = D$$

so C=0, giving the final equation:

$$x(t) = D\cos\left(\sqrt{\frac{k}{m}}t\right)$$

General Differential Equation

Vocabulary:
The ORDER of a differential equation is order of it's highest derivative, so our examples have 1st order and 2nd order.

Note that the dependent variable doesn't have to be *t*.

For example in the equation $y' = xy$ the solution is some function $y = f\left(x\right)$, that is a function where

$f'\left(x\right) = xf\left(x\right)$ for all x in the domain of f or on some constrained interval.

Note that we have solved many differential functions of the form:

$y' = f\left(x\right)$

The solution being $y = \int f\left(x\right)dx + C$

for example:

$y' = x^2$ has the solution $y = \frac{x^3}{3} + C$

Verifying a solution to a differential equation.

While it is not always easy or even possible to solve a differential equation directly, verifying a solution is straight forward.

Example:

If we have the differential equation $y' = \frac{\left(y^2 - 1\right)}{2}$, show that $y(x) = \frac{1 + ce^x}{1 - ce^x}$ is a solution

Note that $y' = \frac{\left(1 - ce^x\right)ce^x - \left(1 + ce^x\right)\left(-ce^x\right)}{\left(1 - ce^x\right)^2} = \frac{2ce^x}{\left(1 - ce^x\right)^2}$

$$\frac{\left(y^2 - 1\right)}{2} = \frac{\left(\left(\frac{1 + ce^x}{1 - ce^x}\right)^2 - 1\right)}{2} = \frac{\left(\frac{\left(1 + ce^x\right)^2 - \left(1 - ce^x\right)^2}{\left(1 - ce^x\right)^2}\right)}{2} = \frac{\left(\frac{4ce^x}{\left(1 - ce^x\right)^2}\right)}{2} = \frac{2ce^x}{\left(1 - ce^x\right)^2}$$

S for every c this solution holds.

Now let's do the initial value problem for this equation where $y(0) = 2$

Plugging in we have $y(0) = \frac{1 + ce^0}{1 - ce^0} = \frac{1 + c}{1 - c} = 2$

So $1 + c = 2 - 2c \rightarrow 3c = 1 \rightarrow c = \frac{1}{3}$

So the solution is $y(x) = \frac{1 + (1/3)e^x}{1 - (1/3)e^x}$