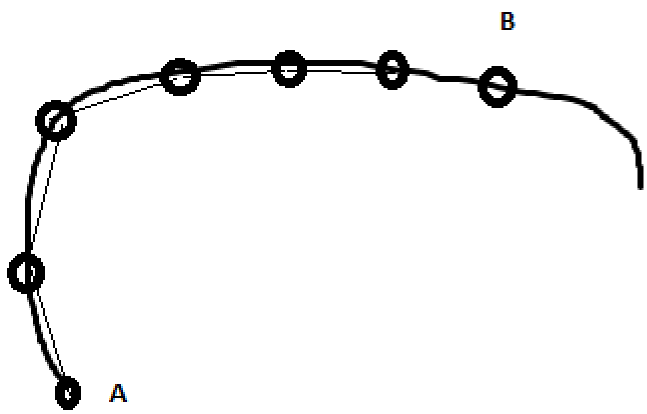
Lesson Plan 14 - Arc Length 6.4+ Avg Value 6.5

1) Take attendance
2) Return MidTerm, Questions?
3) Arc Length

Assume we have a curve described by parametric functions $x = f(t)$ and $y = g(t)$ defined on some interval $a \le t \le b$.

As an approximation we can break the curve up as follows:



Where the points are $A = (x_0, y_0), (x_1, y_1) \ldots (x_n, y_n) = B$

We know that between any two points the length is $L \approx \sum_{i=0}^{n} \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$ where

$\Delta x = x_{i+1} - x_i$ and $\Delta y = y_{i+1} - y_i$

Note however that $f'(t) \approx \dfrac{\Delta x_i}{\Delta t}$ and $g'(t) \approx \dfrac{\Delta y_i}{\Delta t}$

So we can express $\Delta x_i = f'(t)\Delta t$ and $\Delta y_i = g'(t)\Delta t$

Rewriting our sum $L \approx \sum_{i=0}^{n} \sqrt{\left(f'(t_t)\Delta t_i\right)^2 + \left(g'(t_i)\Delta t_i\right)^2}$

Letting the $\Delta t's$ go to zero we get the integral

$$L = \int_a^b \sqrt{\left(f'(t)\right)^2 + \left(g'(t)\right)^2}\, dt \text{ or}$$

$$L = \int_a^b \sqrt{\left(\frac{df}{dt}\right)^2 + \left(\frac{dg}{dt}\right)^2}\, dt$$

Example 1:
Let $x = t^2$ and $y = t^2$ be parametric equations for a curve. What is the length of this curve from $(1,1)$ to $(4,8)$

For $x = 1$, $t = 1$ and for $x = 4$, $t = 2$ so we have the integral:

$$L = \int_1^2 \sqrt{\left(\frac{df}{dt}\right)^2 + \left(\frac{dg}{dt}\right)^2}\, dt = \int_1^2 \sqrt{(2t)^2 + (3t^2)^2}\, dt = \int_1^2 \sqrt{4t^2 + 9t^4}\, dt =$$

$$\int_1^2 t\sqrt{4 + 9t^2}\, dt = \frac{1}{18}\int_1^2 18t\sqrt{4 + 9t^2}\, dt =$$

Subtituting $u = 4 + 9t^2$ we find that $du = 18t$ so

$$L = \frac{1}{18}\int_1^2 18t\sqrt{4 + 9t^2}\, dt = \frac{1}{18}\int \sqrt{u}\ du = \frac{1}{18}\left[\frac{2u^{3/2}}{3}\right] = \frac{1}{27}\left[\left(4 + 9t^2\right)^{3/2}\right]_1^2 =$$

$$\frac{1}{27}\left[(40)^{3/2} - (13)^{3/2}\right] = \frac{1}{27}\left[80\sqrt{10} - 13\sqrt{13}\right]$$

Note that if we are given a function in terms of *x* we can treat *x* as a parameter giving the equations $x = x$ and $y = f(x)$

Since $\frac{dx}{dx} = 1$ our formulae becomes

$$L = \int_a^b \sqrt{\left(\frac{df}{dx}\right)^2 + 1}\ dt$$

Example 2:

Find the length of the arch of the parabola $y^2 = x$ from $(0,0)$ to $(1,1)$

Here we treat y as the parameter so we have

$$L = \int_0^1 \sqrt{\left(\frac{dy^2}{dy}\right)^2 + 1}\, dy = \int_0^1 \sqrt{4y^2 + 1}\, dy =$$

Substitute $u = 2y$ so that $\frac{du}{2} = dy$ giving the integral $\frac{1}{2}\int \sqrt{u^2 + 1}\, du$

From our table #21 we have

$$\int \sqrt{a^2 + u^2}\, du = \frac{u}{2}\sqrt{a^2 + u^2} + \frac{a^2}{2}\ln\left(u + \sqrt{a^2 + u^2}\right) + c$$

$$\frac{1}{2}\int \sqrt{u^2 + 1}\, du = \frac{1}{2}\left[y\sqrt{1 + 4y^2} + \frac{1}{2}\ln\left(2y + \sqrt{4y^2 + 1}\right)\Big|_0^1 \right] =$$

This gives us

$$\frac{1}{2}\left[\sqrt{5} + \frac{\ln\left(2 + \sqrt{5}\right)}{2} - \left(0 + \frac{1}{2}\ln(1)\right)\right] = \frac{\sqrt{5}}{2} + \frac{\ln\left(2 + \sqrt{5}\right)}{4}$$

Try Handout Problems

Average Value of a function.

Let's say you were to take the temperature every hour on the hour for 24 hours getting readings $T_0, T_1, ..., T_{23}$, then the average temperature for the day would be approximately

$$Avg \approx \frac{\sum_{i=0}^{23} T_i}{24}$$

If you were to then increase the readings to 48, 96, etc. You would end up with an integral for the average temperature that looks like this:

$$Avg = \frac{1}{24} \int_{24 hours} T(t) dt$$

We define a more general formula for the average of a function on an interval $[a,b]$ as

$$Avg = \frac{1}{b-a} \int_a^b f(x)\, dx$$

Note and important feature of this. If the function is continuous, then there must some $x = c$ such that $f(c) = Avg$.

Simplistic reason why:

Since the function is bounded there will be a maximum and minimum value of the function. It should be clear that $\min \le Avg \le \max$.

But because of the function is continuous, it must pass through every value of the function between the min and max. So there must be a value c for which $f(c) = Avg$.

This is called the mean value theorem..

Example: for $f(x) = 1 + x^2$ on the interval $[-1, 2]$ find the average value of the function and find a c such that $f(c) = Avg$

First find $\frac{1}{2 - {}^{-}1} \int_{-1}^{2} 1 + x^2 dx = 2$

Then find $1 + c^2 = 2$

Note that in this case, c can be either 1 or -1