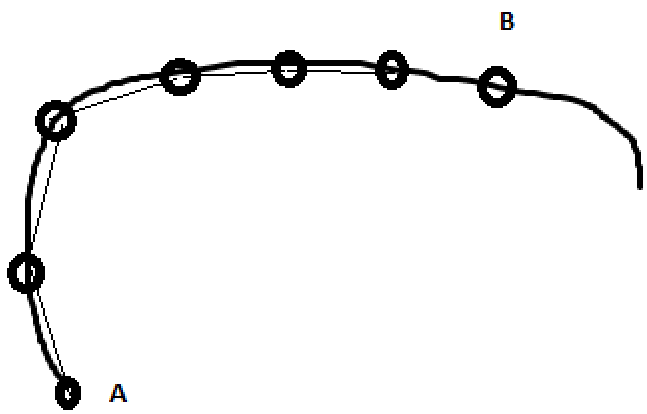
Lesson Plan 9 - 6.5 Length of Curves

1) Take attendance
2) Homework questions?
3) Length of Curves

Assume we have a curve described by parametric functions $x = f(t)$ and $y = g(t)$ defined on some interval $a \le t \le b$.

As an approximation we can break the curve up as follows:



Where the points are $A = (x_0, y_0), (x_1, y_1) \ldots (x_n, y_n) = B$

We know that between any two points the length is $L \approx \sum_{i=0}^{n} \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$ where

$\Delta x = x_{i+1} - x_i$ and $\Delta y = y_{i+1} - y_i$

Note however that $f'(t) \approx \frac{\Delta x_i}{\Delta t}$ and $g'(t) \approx \frac{\Delta y_i}{\Delta t}$

So we can express $\Delta x_i = f'(t)\Delta t$ and $\Delta y_i = g'(t)\Delta t$

Rewriting our sum $L \approx \sum_{i=0}^{n} \sqrt{(f'(t_t)\Delta t_i)^2 + (g'(t_i)\Delta t_i)^2} = \sum_{i=0}^{n} \sqrt{(f'(t_t))^2 + (g'(t_i))^2}\,\Delta t_i$

Letting the $\Delta t's$ go to zero we get the integral

$$L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2}\,dt \text{ or}$$

$$L = \int_a^b \sqrt{\left(\frac{df}{dt}\right)^2 + \left(\frac{dg}{dt}\right)^2}\,dt$$

Example 1:
Let $x = t^2$ and $y = t^2$ be parametric equations for a curve. What is the length of this curve from $(1,1)$ to $(4,8)$

For $x = 1$, $t = 1$ and for $x = 4$, $t = 2$ so we have the integral:

$$L = \int_1^2 \sqrt{\left(\frac{df}{dt}\right)^2 + \left(\frac{dg}{dt}\right)^2}\, dt = \int_1^2 \sqrt{(2t)^2 + (3t^2)^2}\, dt = \int_1^2 \sqrt{4t^2 + 9t^4}\, dt =$$

$$\int_1^2 t\sqrt{4 + 9t^2}\, dt = \frac{1}{18}\int_1^2 18t\sqrt{4 + 9t^2}\, dt =$$

Subtituting $u = 4 + 9t^2$ we find that $du = 18t$ so

$$L = \frac{1}{18}\int_1^2 18t\sqrt{4 + 9t^2}\, dt = \frac{1}{18}\int \sqrt{u}\ du = \frac{1}{18}\left[\frac{2u^{3/2}}{3}\right] = \frac{1}{27}\left[\left(4 + 9t^2\right)^{3/2}\right]_1^2 =$$

$$\frac{1}{27}\left[(40)^{3/2} - (13)^{3/2}\right] = \frac{1}{27}\left[80\sqrt{10} - 13\sqrt{13}\right]$$

Note that if we are given a function in terms of *x* we can treat *x* as a parameter giving the equations $x = x$ and $y = f(x)$

Since $\frac{dx}{dx} = 1$ our formulae becomes

$$L = \int_a^b \sqrt{\left(\frac{df}{dx}\right)^2 + 1}\ dt$$

Example 2:

Find the length of the arch of the parabola $y^2 = x$ from $(0,0)$ to $(1,1)$

Here we treat y as the parameter so we have

$$L = \int_0^1 \sqrt{\left(\frac{dy^2}{dy}\right)^2 + 1}\, dy = \int_0^1 \sqrt{4y^2+1}\, dy =$$

Substitute $u = 2y$ so that $\frac{du}{2} = dy$ giving the integral $\frac{1}{2}\int \sqrt{u^2+1}\, du$

From our table #21 we have

$$\int \sqrt{a^2+u^2}\, du = \frac{u}{2}\sqrt{a^2+u^2} + \frac{a^2}{2}\ln\left(u + \sqrt{a^2+u^2}\right) + c$$

$$\frac{1}{2}\int \sqrt{u^2+1}\, du = \frac{1}{2}\left[y\sqrt{1+4y^2} + \frac{1}{2}\ln\left(2y + \sqrt{4y^2+1}\right)\Big|_0^1 \right] =$$

This gives us

$$\frac{1}{2}\left[\sqrt{5} + \frac{\ln\left(2+\sqrt{5}\right)}{2} - \left(0 + \frac{1}{2}\ln(1)\right)\right] = \frac{\sqrt{5}}{2} + \frac{\ln\left(2+\sqrt{5}\right)}{4}$$

Example 3:

What is the arc length of the curve $f(x) = 2e^x + \frac{1}{8}e^{-x}$ on the interval $[0, \ln 2]$

First we find $f'(x) = 2e^x - \frac{1}{8}e^{-x}$

and

$$[f'(x)]^2 = 4e^{2x} - \frac{1}{2} + \frac{1}{64}e^{-2x}$$

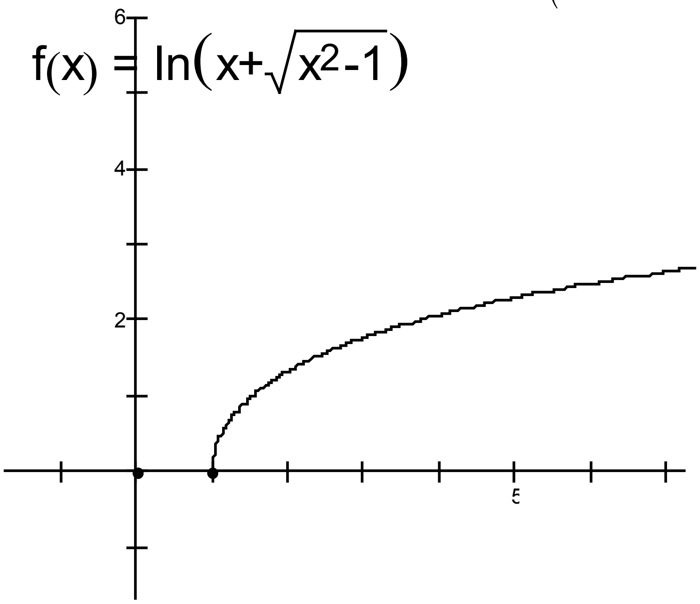
so

$$L = \int_0^{\ln 2} \sqrt{1 + 4e^{2x} - \frac{1}{2} + \frac{1}{64}e^{-2x}}\, dx = \int_0^{\ln 2} \sqrt{4e^{2x} + \frac{1}{2} + \frac{1}{64}e^{-2x}}\, dx = \int_0^{\ln 2} \sqrt{\left(2e^x + \frac{1}{8}e^{-x}\right)^2}\, dx =$$

$$\int_0^{\ln 2} 2e^x + \frac{1}{8}e^{-x} dx = \left[2e^x - \frac{1}{8}e^{-x}\right]_0^{\ln 2} = 4 - \frac{1}{16} - \left(2 - \frac{1}{8}\right) = 2 + \frac{1}{16} = \frac{33}{16}$$

Example 4: (Hard)

Find the length of the curve $y = f(x) = \ln\left(x + \sqrt{x^2 - 1}\right)$ on the interval $\left[1, \sqrt{2}\right]$



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The problem with this function is that at 1, the tangent is vertical and therefore $f'(x)$ is undefined.

We note however that $f(x)$ is actually the $\cosh^{-1}(x) = \dfrac{e^x + e^{-x}}{2}$

Proof:

$y = \ln\left(x + \sqrt{x^2 - 1}\right)$

$e^y = x + \sqrt{x^2 - 1}$

$e^y - x = \sqrt{x^2 - 1}$

squaring both sides gives us

$e^{2y} - 2xe^y + x^2 = x^2 - 1$

$e^{2y} - 2xe^y = -1$

$2xe^y = e^{2y} + 1$

$x = \dfrac{e^y + e^{-y}}{2} = \cosh(y)$

If $x \in \left[1, \sqrt{2}\right]$ then $y \in \left[0, \ln\left(\sqrt{2} + 1\right)\right]$

So

$$L = \int_0^{\ln(\sqrt{2}+1)} \sqrt{1-[\cosh' y]^2}\,dy = \int_0^{\ln(\sqrt{2}+1)} \sqrt{1-(\sinh y)^2}\,dy = \int_0^{\ln(\sqrt{2}+1)} \sqrt{(\cosh y)^2}\,dy = \int_0^{\ln(\sqrt{2}+1)} \cosh y\, dy =$$

$$[\sinh y]_0^{\ln(\sqrt{2}+1)} = \sinh\left(\ln\left(\sqrt{2}+1\right)\right) - \sinh(0) = \frac{e^{ln(\sqrt{2}+1)} - e^{-ln(\sqrt{2}+1)}}{2} - \frac{e^0 - e^{-0}}{2} =$$

$$\frac{\sqrt{2}+1-1/\sqrt{2}+1}{2} = \frac{2+2\sqrt{2}+1-1}{2\left(\sqrt{2}+1\right)} = \frac{1+\sqrt{2}}{1+\sqrt{2}} = 1$$

Try Handout Problems