Lesson Plan 4 - Working with Integrals 5.4

1) Take attendance
2) Quiz next Tuesday
3) Questions on the worksheet?
4) Go over the homework from last week

Let's review what even and odd functions are.

Definition:

If $f(-x) = f(x)$, then $f(x)$ is an even function.

Examples:

$f(x) = C$

$f(x) = x^2$

$f(x) = |x|$

$f(x) = \cos x$

What does the graph of these functions have in common?

If $f(-x) = -f(x)$, then $f(x)$ is an odd function.

Examples:

$f(x) = 0$

$f(x) = Cx$

$f(x) = x^3$

$f(x) = \sin x$

$f(x) = \tan x$

What does the graph of these functions have in common?

Note that a function does not have to be even or odd, eg.

$f(x) = x + 5$

Some other observations:

If $f(x)$ is an odd function and $g(x)$ is an even function then $f(x)g(x)$ and $f(x)/g(x)$ are both odd functions.

Also note that if $f(x)$ is an odd function, $|f(x)|$ must be even.

Note that by symmetry, if $f(x)$ is an even function defined on $[-a,a]$, then we have

$$\int_{-a}^{0} f(x)\,dx = \int_{0}^{a} f(x)\,dx$$

and therefore.

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

Note that by symmetry, if $f(x)$ is an odd function defined on $[-a,a]$, then we have

$$\int_{-a}^{0} f(x)\,dx = -\int_{0}^{a} f(x)\,dx$$

and therefore.

$$\int_{-a}^{a} f(x)\,dx = 0$$

It is sometimes useful to note the even-ness or odd-ness of a function when integrating, for example:

$\int_{-\pi}^{\pi} \frac{e^{x^3}}{x^9}\,dx = 0$, WHY?

Example 1:

$$\int_{-2}^{2}\left(x^4-3x^3\right)dx=\int_{-2}^{2}x^4\ dx+3\int_{-2}^{2}x^3\ dx=2\int_{0}^{2}x^4\ dx+0=2\left[\frac{x^5}{5}\right]_0^2=\frac{64}{5}$$

Example 2:

$$\int_{-\pi/2}^{\pi/2}\left(\cos x-4\sin^3 x\right)dx=\int_{-\pi/2}^{\pi/2}\cos x\ dx-\int_{-\pi/2}^{\pi/2}-4\sin^3 x\ dx=\int_{-\pi/2}^{\pi/2}\cos x\ dx=\left[\sin x\right]_{-\pi/2}^{\pi/2}=1-{}^{-}1=2$$

Average Value of a Function

We are all familiar with discrete averages. For example if we were to tally the weight of each member of class and divide by the number of students, we would get the average weight.

What about a road trip in which the speed of our car varies. At the end of the trip we could divide how far we've traveled by the time it took. But if you look at a velocity graph of the trip, the distance we've traveled is the area under the curve. We can generalize this to define the average value of a function as:

$$\overline{f}=\frac{\int_a^b f(x)\,dx}{b-a}$$

Example 3

A hiking trail has an elevation given by

$$f(x)=60-x^3-650x^2+1200x+4500$$

What is the average elevation between 0 and 5?

$$\frac{\int_0^5 f(x)\,dx}{5-0}=\frac{1}{5}\int_0^5\left(60x^3-650x^2+1200x+4500\right)dx$$

Mean value Theorem for Integrals

Note that for a continuous (smooth) function, at it's minimum, the function will be less than or equal to the average value, and at it's maximum value it will be greater than or equal to it's average value.

Since the function is smooth it will pass through every value in between the maximum and minimum. This brings us to an important theorem in analysis called

The Mean value theorem for integrals. It goes like this.

If $f(x)$ is a continuous function on $[a,b]$ then there exists a value $c \in [a,b]$ such that

$$f(c) = \frac{\int_a^b f(x)\,dx}{b-a}$$

Example 4:

Find the point(s0 on the interval $(0,1)$ where $f(x) = 2x(1-x)$ equals the average value on $[0,1]$

$$\overline{f} = \frac{1}{1-0}\int_0^1 \left(2x(1-x)\right)\,dx = \left[x^2 - \frac{2x^3}{3}\right]_0^1 = 1 - \frac{2}{3} = \frac{1}{3}$$

Hand out Work Sheet and go Over