

Lesson Plan 3 - The Fundamental Theorem of Calculus 5.3

- 1) Take attendance
- 2) Quiz next Tuesday
- 3) Questions on the homework or the work sheet?

We have previously discussed the very important connection between Differential and Integral Calculus known as **The Fundamental Theorem of Calculus**.

In fact, calculating definite integrals and evaluating indefinite integrals would be very difficult without this connection.

There are a number of forms of this theorem but one version divides it into two parts:

For both parts we assume that f is continuous on some interval $[a, b]$, and $a \leq x \leq b$.

Part 1

Let $F(x) = \int_a^x f(t) dt$, then

$$\frac{d}{dx} F(x) = f(x)$$

This tells us that if $F(x)$ is a function that calculates the area under a function $f(x)$ between some constant a and x , then the derivative with respect to x of $F(x)$ is $f(x)$.

Part 2

If $\frac{d}{dx} F(x) = f(x)$ then $\int_a^b f(x) dx = F(b) - F(a)$

This tells us that if $F(x)$ is an anti-derivative of $f(x)$, then we can calculate the area under $f(x)$ on the interval $[a, b]$ using $F(x)$.

A note about combining the chain rule with the fundamental theorem
(Example 5, page 370)

$\frac{d}{dx} \int_a^{x^4} \sin(t) dt$ In this case we must be careful since our endpoint is a function of the independent variable. To calculate this properly we must invoke the chain rule as follows:

First set $u = x^4$

Then our integral becomes $\frac{d}{dx} \left[\int_a^u \sin(t) dt \right] \frac{du}{dx} = \sin(u) \frac{du}{dx} = \sin(x^4) \cdot 4x^3$

Using the Fundamental Theorem

Some Useful Formulas for finding anti-derivatives (**Indefinite Integrals**):

$$1) \int x^n dx = \frac{x^{n+1}}{n+1} + C \text{ for } x \neq -1$$

$$2) \int \frac{1}{x} dx = \ln|x| + C$$

$$3) \int e^x dx = e^x + C$$

$$4) \int \cos(x) dx = \sin(x) + C$$

$$5) \int \sin(x) dx = -\cos(x) + C$$

Note the following important pattern:

$$\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C \quad \text{This comes up a lot, eg.}$$

$$\int \frac{2x+1}{x^2+x+4} dx$$

$$\text{Since } \frac{d}{dx}(x^2+x+4) = 2x+1$$

$$\int \frac{2x+1}{x^2+x+4} dx = \ln|x^2+x+4| + C$$

An enhancement on this formula

$$\int \frac{f'(x)}{f(x)^n} dx = -\left(\frac{1}{n-1}\right) \frac{1}{f(x)^{n-1}} + C$$

Example:

$$\int \frac{x}{(x^2+1)^2} dx$$

We know that $\frac{d}{dx}(x^2+1) = 2x$

$$\text{So we have } \int \frac{x}{(x^2+1)^2} dx = \frac{1}{2} \int \frac{2x}{(x^2+1)^2} dx = \frac{1}{2} \left(-\frac{1}{1}\right) \frac{1}{x^2+1} + C = \left(-\frac{1}{2}\right) \frac{1}{x^2+1}$$

Example, find: $\int \sec(x) \tan(x) dx$

$$\int \sec(x) \tan(x) dx = \int \frac{1}{\cos(x)} \cdot \frac{\sin(x)}{\cos(x)} dx = \int \frac{\sin(x)}{\cos(x)^2} dx = -\int \frac{-\sin(x)}{\cos(x)^2} dx$$

But $[\cos(x)]' = -\sin(x)$

$$\text{So we have } \int \sec(x) \tan(x) dx = -\left(-\frac{1}{\cos(x)}\right) + C = \frac{1}{\cos(x)} + C = \sec(x) + C$$

Another Pattern

$$\text{Recall that } \frac{d}{dx} \sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$$

How do we know this?

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

$$\text{If } x = \sin(y) \text{ then } \frac{dx}{dy} = \cos(y)$$

$$\frac{dy}{dx} = \frac{1}{\cos(y)} = \frac{1}{\sqrt{1-\sin^2(y)}} = \frac{1}{\sqrt{1-x^2}}$$

$$\text{So clearly we have } \int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1}(x) + C$$

$$\text{But likewise } \int \frac{f'(x)}{\sqrt{1-f(x)^2}} dx = \sin^{-1}(f(x)) + C$$

$$\text{Example: } \int \frac{x}{\sqrt{1-x^4}} dx$$

$$\text{Since } \frac{d}{dx} x^2 = 2x$$

$$\int \frac{x}{\sqrt{1-x^4}} dx = \frac{1}{2} \int \frac{2x}{\sqrt{1-(x^2)^2}} dx = \frac{1}{2} \sin^{-1}(x^2) + C$$

A similarly useful formula

$$\int \frac{f'(x)}{1+f(x)^2} dx = \tan^{-1}(f(x)) + C$$

Simplifying before integrating

$$\int_1^9 \frac{2t^2 + t^2 \sqrt{t} - 1}{t^2} dx = \int_1^9 2 + t^{1/2} - \frac{1}{t^2} dx = \left[2t + \frac{2}{3} t^{3/2} + \frac{1}{t} \right]_1^9 =$$
$$18 + \frac{2}{3} \cdot 27 + \frac{1}{9} - \left(2 + \frac{2}{3} + 1 \right) = 36 - 3 + \frac{1}{9} - \frac{2}{3} = 33 + \frac{1-6}{9} = 33 - \frac{5}{9} = 32 \frac{4}{9}$$

Give Students Handout 3 with some examples to try:

Net Change

We can rewrite

$$\int_a^b f(x)dx = F(b) - F(a)$$

as

$$\int_a^b F'(x)dx = F(b) - F(a)$$

Where $F'(x)$ is the rate of change of $F(x)$

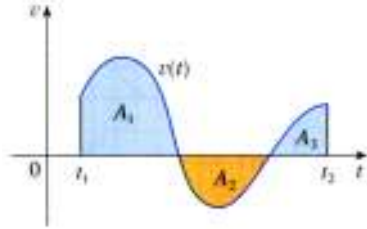
Example:

Consider a reservoir which has water flowing into or out of it at a rate of $V'(t)$

So $V(t_2) - V(t_1) = \int_{t_1}^{t_2} V'(t)dt$ is the change in the amount of water between time t_1 and t_2 .

Displacement vs. distance traveled.

Consider a train that travels back and forth on a straight rail at a velocity according to this graph.



If you wish to know its displacement then

$$\text{displacement} = A_1 - A_2 + A_3 = \int_{t_1}^{t_2} v(t) dt$$

If instead you want to know the distance traveled then you want

$$\text{distance} = A_1 + A_2 + A_3 = \int_{t_1}^{t_2} |v(t)| dt$$

Example from the book:

A particle has velocity $v(t) = t^2 - t - 6$

Find the displacement and distance traveled between 1 and 4 seconds:

$$\text{displacement} = \int_1^4 (t^2 - t - 6) dt =$$

$$\left[\frac{t^3}{3} - \frac{t^2}{2} - 6t \right]_1^4 = \frac{64}{3} - 8 - 24 - \left(\frac{1}{3} - \frac{1}{2} - 6 \right) = -\frac{9}{2}$$

$$\text{distance} = \int_1^4 |t^2 - t - 6| dt$$

Find where the direction changes: $t^2 - t - 6 = (t - 3)(t + 2) = 0$

So it changes at -2 and 3 seconds.

$$\int_1^4 |t^2 - t - 6| dt = \int_1^3 |t^2 - t - 6| dt + \int_3^4 |t^2 - t - 6| dt =$$

$$\left| \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t \right]_1^3 \right| + \left| \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t \right]_3^4 \right| =$$

$$\left| 9 - \frac{9}{2} - 18 - \left(\frac{1}{3} - \frac{1}{2} - 6 \right) \right| + \left| \frac{64}{3} - 8 - 24 - \left(9 - \frac{9}{2} - 18 \right) \right| =$$

$$\left| -\frac{22}{3} \right| + \left| \frac{17}{6} \right| = \frac{44}{6} + \frac{17}{6} = \frac{61}{6}$$

A good time to break

Give Students Class Handout 3 to work on and go over in class.

