Lesson Plan 18 - Numerical Integration 7.7

1) Take attendance
2) Quiz on Tuesday
3) Bring Picture ID to FINAL!

Why do we need to learn about Approximate integration.

First, some integrals do not have an anti-derivative.

The classic example is $\int e^{-x^2}\,dx$.

There are many others, but this integral is important in statistics and there very important for business and the social sciences.

However it is very easy to evaluate this as a definite integral, for example on a Graphic calculator.

So why would we need to do an approximate integration?

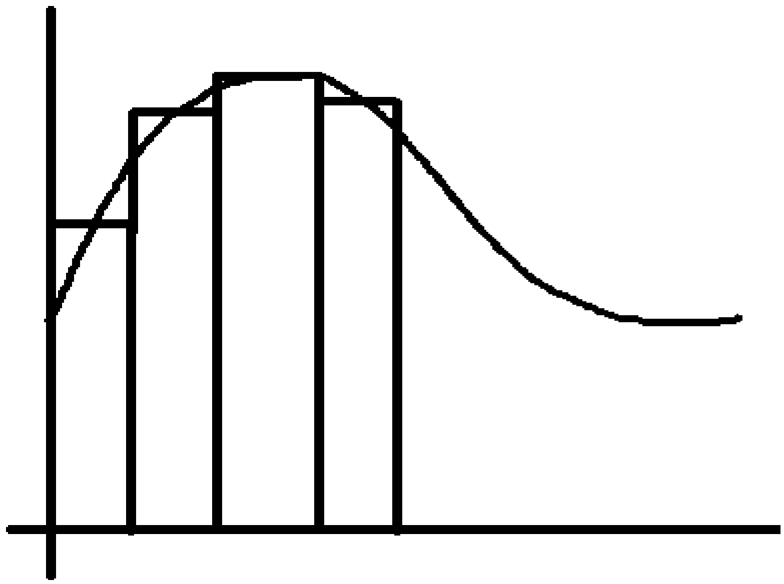
If you are computer programmer and you need to write some code like the code your graphics calculator uses, you will need to know more about how it is done.

Also, if you are collecting data points for a function, and you want to integrate that function, then you will need one of the methods we will describe because you do not have a function.

The first method we will look at is using the MIDPOINT rule.

Basically we break up the interval we are integrating on and find the area of rectangles whose height is at the middle of the of each interval.

Midpoint Rule:



The formula looks like this:

$$A = \Delta x \sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right) \text{ where } \Delta x = \frac{x_n - x_0}{n-1}$$

This is the least accurate way we will look at, and it has a distinct disadvantage, that we might not know the value of the function at the midpoint.

As an example, we use the midpoint rule to find the integral:

$\int_2^{10} x^2 dx$ using 4 sub-intervals

The points then are $\{3,5,7,9\}$ with $\Delta x = (10-2)/4 = 2$

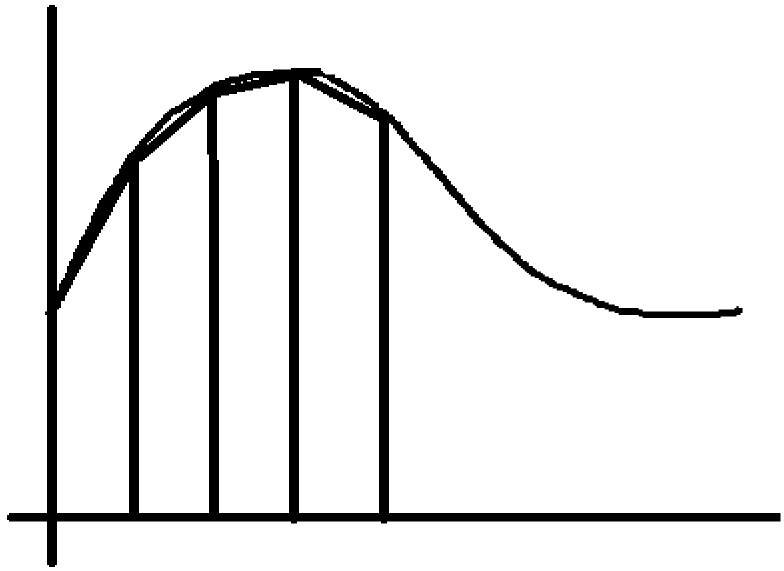
$$\int_2^{10} x^2 dx \simeq 2\left[3^2+5^2+7^2+9^2\right] = 2\left[9+25+49+81\right] = 2(144) = 288$$

$$\int_2^{10} x^2 dx = \left[\frac{x^3}{3}\right]_2^{10} = \frac{1000-8}{3} = \frac{992}{3}$$

The error here is then $\frac{128}{3}$ and the relative error is $\frac{\frac{128}{3}}{\frac{992}{3}} \simeq 13\%$

A somewhat more accurate method is the TRAPEZOIDAL Rule in which we break up the area into trapezoids.

Trapezoidal Rule:



Area of a trapezoid is $A = \frac{w_1 + w_2}{2} h$ so we get

$$A = \Delta x \sum_{i=0}^{n-1} \frac{f(x_i) + f(x_{i+1})}{2} \quad \text{where } \Delta x = \frac{x_n - x_0}{n-1}$$

Let's try the same problem using this rule.

The points are $\{2, 4, 6, 8, 10\}$

$$\int_2^{10} x^2 dx \simeq 2\left[\frac{2^2+4^2}{2} + \frac{4^2+6^2}{2} + \frac{6^2+8^2}{2} + \frac{8^2+10^2}{2}\right] =$$

$$4+16+16+36+36+64+64+100 = 336$$

The error here is then $\frac{16}{3}$ and the relative error is $\frac{\frac{16}{3}}{\frac{992}{3}} \simeq 1.7\%$

What can we conclude looking at these answers?

In this case, the midpoint is better than Trapezoid.

Error Bounds

There is a way of estimating a maximum error for Midpoint and Trapezoid.

Suppose $\left|f''(x)\right| \leq K \ \ for\ a \leq x \leq b$

Then the errors E_T and E_M can be bounded by

$\left|E_T\right| \leq \dfrac{K(b-a)^3}{12n^2}$ and $\left|E_M\right| \leq \dfrac{K(b-a)^3}{24n^2}$

This indicates that the maximum error when using the Trapezoidal rule is twice that when using the Midpoint rule.

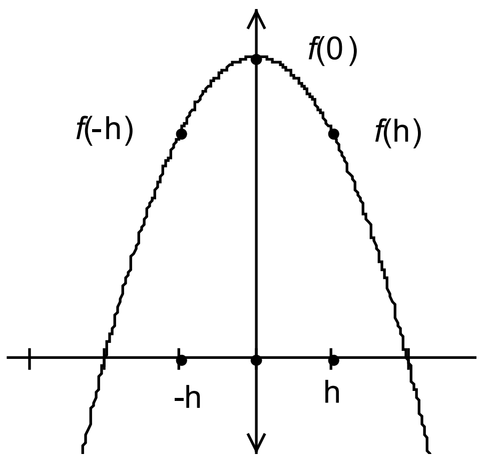
Why would is this useful?

If you want to write a computer code that runs on a calculator, you want it to run as fast as possible. So you want to know at what point your calculation would exceed the accuracy of numbers used.

A better approximation called Simpson's rule approximates the function between each group of three points using a parabola. The resulting sum is surprisingly simple:

$$A_{parabola} = \frac{f(x_0) + 4f(x_1) + f(x_2)}{3}\Delta x$$

This is suspiciously simple and we should verify it.



Consider the parabolic equation and the area underneath it

$$Area = \int_{-h}^{h} Ax^2 + Bx + C\ dx = \int_{-h}^{h} Ax^2 + C\ dx + \int_{-h}^{h} Bx\ dx$$

Note that the first term is an even function and the 2nd term is an odd function. So we may rewrite this as

$$Area = 2\int_{0}^{h} Ax^2 + C\ dx + 0 = 2\left[\frac{Ax^3}{3} + Cx\right]_0^h = 2\left(\frac{Ah^3}{3} + Ch\right) = \frac{h}{3}\left(2Ah^2 + 6C\right)$$

Now note what happens when we add

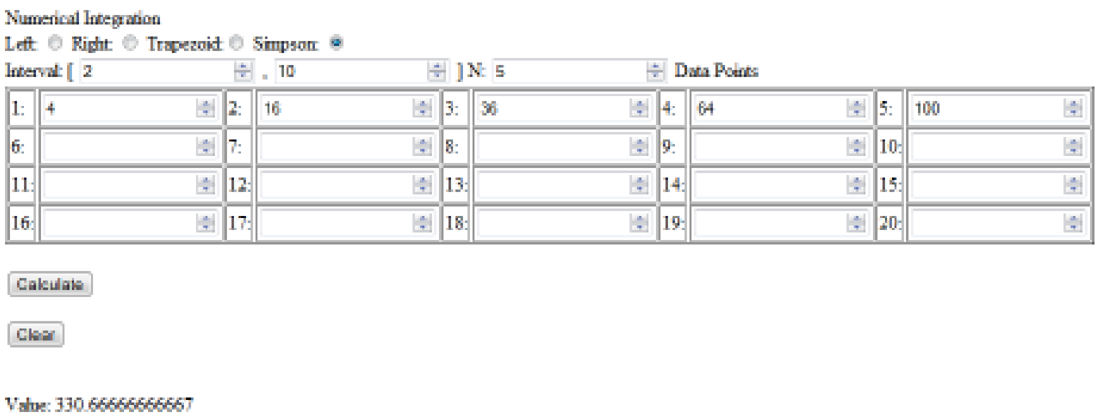
$f(-h) + 4f(0) + f(h) = Ah^2 - Bh + C + 4C + Ah^2 + Bh + C = 2Ah^2 + 6C$

So $Area = \frac{h}{3}\left(f(-h) + 4f(0) + f(h)\right)$

We can translate this parabola left and right to any three points, so we can sum as follows:

$$Area = \frac{\Delta x}{3}\sum_{i=0}^{i=}\left(f(x_{i*3}) + 4f(x_{i*3+1}) + f(x_{i*3+2})\right)$$

Let's now try the website calculator again:
http://foothill.schoenbrun.com/math1b-2/integrate.php



Note the error for Simpson's rule is given by:

$$.000001 = \left|E_S\right| \le \frac{K\left(b-a\right)^5}{180n^4} = .000017$$

In this particular case, the error is essentially zero, so Simpson's rule is generally a superior method.