Lesson Plan 17 - Linear Differential Equations and Vector Spaces and Orthogonal Tragectories

Linear Differential Equations

Consider a generic linear differential equation:

$$a_n y^{(n)'} + a_{n-1} y^{(n-1)'} + \ldots + a_2 y^{''} + a_1 y^{'} + a_0 y = 0$$

We can try as a possible solution $y = e^{rx}$.

So $y^{(i)'} = r^i e^{rx}$ and

$$a_n y^{(n)'} + a_{n-1} y^{(n-1)'} + \ldots + a_2 y^{''} + a_1 y^{'} + a_0 y =$$
$$a_n r^n e^{rx} + a_{n-1} r^{n-1} e^{rx} + \ldots + a_2 r^2 e^{rx} + a_1 r e^{rx} + a_0 e^{rx} = 0$$

Multiplying through by e^{-rx} we get

$$a_n r^n + a_{n-1} r^{n-1} + \ldots + a_2 r^2 + a_1 r + a_0 = 0$$

This is an *n*'th degree polynomial equation with variable r which will have *n* roots, $\{r_1, r_2, \ldots, r_n\}$

So for each such root, $y = e^{r_i x}$ will be a solution.

Linearity

Note that if we have some solution to the original equation y then

$$a_n Cy^{(n)'} + \ldots + a_0 Cy =$$
$$C\left[a_n y^{(n)'} + \ldots + a_0 y\right] = C[0] = 0$$

so
Cy is also a solution for any C.

Likewise if we have two solutions y and z of a differential equation then

$$a_n\left(y^{(n)'} + z^{(n)'}\right) + \ldots + a_0\left(y + z\right) =$$
$$\left[a_n y^{(n)'} + \ldots + a_0 y\right] + \left[a_n z^{(n)'} + \ldots + a_0 z\right] = 0 + 0 = 0$$

so $y+z$ is also a solution.

Therefore for our example equation, $\sum C_i e^{r_i x}$ will be a solution.

Differential equations that are linear have this property, that a linear combination of solutions is always a solution.

If n=1 we have the simple equation $y' + ay = 0$ with $r_1 = a$

so the general solution is $y = Ce^{-ax}$

For a 2nd order equation, $y'' + ay' + by = 0$

if $y = e^{rx}$ then $y' = re^{rx}$ and $y'' = r^2 e^{rx}$

So the equation becomes $r^2 e^{rx} + are^{rx} + be^{rx} = 0$

If we multiply through again by e^{-rx} we get
$r^2 + ar + b = 0$ a quadratic equation whose solutions are

$$r_1 = \frac{-a + \sqrt{a^2 - 4b}}{2} \text{ and } r_1 = \frac{-a - \sqrt{a^2 - 4b}}{2}$$

This should look familiar. Looking at the discriminant

if $a^2 = 4b$ the solution simplifies to $y = e^{(-a/2)x}$.

If $a^2 > 4b$ then we get two solutions

with $r_1 = \dfrac{-a + \sqrt{a^2 - 4b}}{2}$ and $r_1 = \dfrac{-a - \sqrt{a^2 - 4b}}{2}$

we have $y = e^{r_1 x}$ and $y = e^{r_2 x}$ and so the general solution is

$$y = Ae^{r_1 x} + Be^{r_2 x}$$

Finally if $a^2 < 4b$ the roots are complex and we end up with solutions:

$$y = Ae^{-ax}\cos\beta + Be^{-ax}\sin\beta \text{ where } \beta = \frac{\sqrt{4b - a^2}}{2}$$

Example:

$$y'' + 5y' + 4y = 0 \text{ where } y(0) = 0 \text{ and } y'(0) = 1$$

Since $5^2 - 4 \cdot 4 = 9 > 0$ $r_1 = \dfrac{-5 + \sqrt{9}}{2} = -1$ and $r_2 = \dfrac{-5 - \sqrt{9}}{2} = -4$

so $y = Ae^{-x} + Be^{-4x}$ and $y' = -Ae^{-x} + -4Be^{-4x}$

The initial conditions tell us that $y(0) = 0 = A + B$ and $y'(0) = 1 = -A - 4B$

Solving these two equations we get $A = -\dfrac{1}{3}$ and $B = \dfrac{1}{3}$ so the final solution is

$$y = \frac{-e^{-x} + e^{-4x}}{3}$$

Summary

$$\sum_{i=0}^{n} a_i y^{'(i)} = 0$$

there is an associated polynomial equation

$\sum_{i=0}^{n} a_i r^i = 0$ whose roots $r = \{r_1, r_2, \ldots, r_n\}$ have the property that

$y = e^{r_i x}$ is a solution and therefore $y = \sum_{i=1}^{n} A_i e^{r_i x}$ is a solution.

A question was left over, what happens when there are duplicate roots?

I suggested that for an equation such as:

$y'' - 2y' + y = 0$ that $y = xe^{rx}$ might also be a solution. Let's see:

The associated polynomial here is $r^2 - 2r + 1 = 0 \rightarrow (r-1)^2 = 0$
so r=1, but it is a duplicated root.

We know that $y = e^x$ must be a solution, but let's check $y = xe^x$

$$y' = xe^x + e^x$$
$$y'' = xe^x + 2e^x$$
$$y'' - 2y' + y = xe^x + 2e^x - 2\left[xe^x + e^x\right] + xe^x = xe^x + 2e^x - 2xe^x - 2e^x + xe^x = 0$$

This is highly suggestive that when a differential equation's associated polynomial has *n* duplicated roots that solutions:

$y = x^i e^{rx}$ where $i \in \{0, 1, \ldots n\}$

This suggests something quite beautiful about these equations, but to explain this, I will have to digress and talk about a pre-calculus subject, Vectors.

First lets do a few problems: Give out handout?

Vectors and Vector Spaces

You may recall that a vector is a directed line segment or arrow with a head and a tail. We add vectors by positioning the tail of one vector at the head of the second, with the sum being the vector starting at the tail of the first vector and ending at the tail of the 2nd vector. So we have the property that if

1) $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$ are vectors, then $\overrightarrow{v_1}+\overrightarrow{v_2}$ is a vector.
2) Also we have that if $a \in \mathbb{R}$ and $\vec{v}$ is a vector then $\overrightarrow{av}$ is a vector.
3) There is a special vector called the zero vector which is written $\vec{0}$ which has the property that if $\vec{v}$ is a vector then $\vec{v}+\vec{0}=\vec{v}$.
4) We also have a distributive property $a\left(\overrightarrow{v_1}+\overrightarrow{v_2}\right)=a\overrightarrow{v_1}+a\overrightarrow{v_2}$

When an objects such as a 3D Euclidean vectors have these three properties they are called a VECTOR SPACE.

There are other mathematical objects besides 3D Euclidean vectors that can form a Vector Space.

Before we go on, there are a few properties of a vector space that we need to explore.

In the case of the 3D Euclidean vector space we have defined three vectors $\vec{i},\vec{j},\vec{k}$ which are called unit vectors in the *x, y* and *z* directions respectively.

The important point here is not the length of a unit vector which is one, nor that they are perpendicular, but that they are linearly independent. By that I mean there are no constants A,B such that $\vec{i}=A\vec{j}+B\vec{k}$ nor $\vec{j}=A\vec{i}+B\vec{k}$ nor $\vec{k}=A\vec{i}+B\vec{j}$

They also have the property that any vector can be expressed as

$\vec{V}=a\vec{i}+b\vec{j}+c\vec{k}$ That is they SPAN the vector space.

A property of all vector spaces is that they will have a set of vectors such as these that are both linearly independent and can span the vector space. Such as set is called a BASIS. A basis is not unique. Consider the set $\left\{2\vec{i},2\vec{j},2\vec{k}\right\}$ which is a basis for the same vector space as $\left\{\vec{i},\vec{j},\vec{k}\right\}$.

The property that all bases of a vector space have in common is that they have the same number of elements. This number is called the DIMENSION of the vector space.

The meaning of dimension might seem obvious in a 3D Euclidean vector space, however it is not quite so obvious in other contexts.

Back to linear differential equations

It would seem then that the solutions to a linear differential equation might form a vector space.

Let's check the properties.

We've already seen that if $f(x)$ and $g(x)$ are solutions then we have $Af(x)+Bg(x)$ must also be a solution.

If $y=0$ then all it's derivatives are zero so it will always be a solution to every linear differential equation.

It should now be clear how to construct a basis for such a vector space.

1) Find the roots of the associated polynomial.
By the fundamental theorem of Algebra we know there will be *n* where *n* is the degree of the associated polynomial as well as the order of the differential equation.
2) If a root r is singular, then $x=e^{rx}$ is a solution and a basis member.
3) If a root r is non-singular, duplicated *n* times, then $\{e^{rx}, xe^{rx}, \cdots, x^n e^{rx}\}$ are all solutions.
Since each of these solutions is linearly independent, they form a basis, and there therefore they span the vector space of solutions. That is every possible solution is a linear combination of these solutions.

So it seems that every nth order linear differential equation has a solution set which forms a vector space.

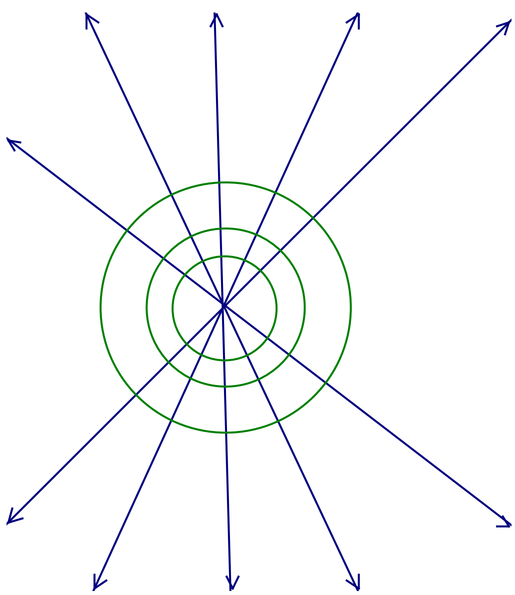
Orthogonal Trajectories

First off, please give me a definition of ORTHOGONAL?

An orthogonal trajectory is a family of curves that intersect each curve of another family orthogonally. that is at right angles.

The simple example would be $y=b$ is an orthogonal trajectory for $x=a$.

Another simple example would be $x^2 + y^2 = R^2$, the set of circles centered at (0,0) and $y = ax$ the set of lines through the origin.



Example:

What about the family of curves $x = ky^2 \rightarrow \frac{dy}{dx} = \frac{y^2}{2xy} = \frac{y}{2x}$.

What does this family look like?

To find an orthogonal trajectory for this family, we first find a differential equation that is satisfied by all members of the family.

First we differential implicitly.

$$\frac{d}{dx}x = \frac{d}{dx}ky^2 \rightarrow 1 = 2ky\frac{dy}{dx} \text{ so } \frac{dy}{dx} = \frac{1}{2ky}.$$

To eliminate the *k* we take the original equation and solve for *k* and then substitute.

$$k = \frac{x}{y^2} \rightarrow \frac{dy}{dx} = \frac{y^2}{2xy} = \frac{y}{2x}$$

To be an orthogonal trajectory the slopes of the curves must be the negative reciprocal so it must be a solution to

$$\frac{dy}{dx} = -\frac{2x}{y}$$

We solve this by separating the variables

$$\int y\,dy = -\int 2x\,dx \rightarrow \frac{y^2}{2} = -x^2 + C \rightarrow \frac{y^2}{2C} + \frac{x^2}{C} = 1$$

This is the equation of an ellipse with major axis *C* along *y* and minor axis *C*/2 along the *x* axis.

Let's try some examples from the handout.