Lesson Plan 16 - Differential Equations 7.9

Differential Equations

What is a Differential equation.

It is an equation that includes an unknown function and one more of its derivatives.

An example might be:

$f(x) = Cf'(x)$

We will typically write this as $y = Cy'$ or $y = C\frac{dy}{dx}$

Often the solution of a differential equation will be a family of solutions.
If the problem to be solved has some initial conditions, this may narrow the solution down to a single function.

Because the function e^x has the unusual property that $\frac{d}{dx}e^x = e^x$ it plays an important role in solving differential equations that we will see as we go forward.

We start by looking at a specific example using the growth of a population. This could be a population of people, animals or bacterial. We assume that the rate of increase in a population is related to the size of the population. We write this as follows:

$\frac{dP}{dt} = kP$ where k is some constant.

This means that the rate of change of a population is a multiple of the current population.

We rewrite this as

$k = \frac{P(t)'}{P(t)}$ and then integrate both sides.

This gives us $\int k \, dt = \int \frac{P(t)'}{P(t)} dt$ and simplifying we find that

$kt + C = \ln|P(t)|$ with C a contant.

Since population is always positive we can drop the absolute value, and we solve for $P(t)$

$P(t) = e^{kt+C} = Ae^{kt}$

Note the role that e^x plays here.

One useful property of differential equations is that while it is sometimes difficult to solve them, verifying them is usually quite straight forward.

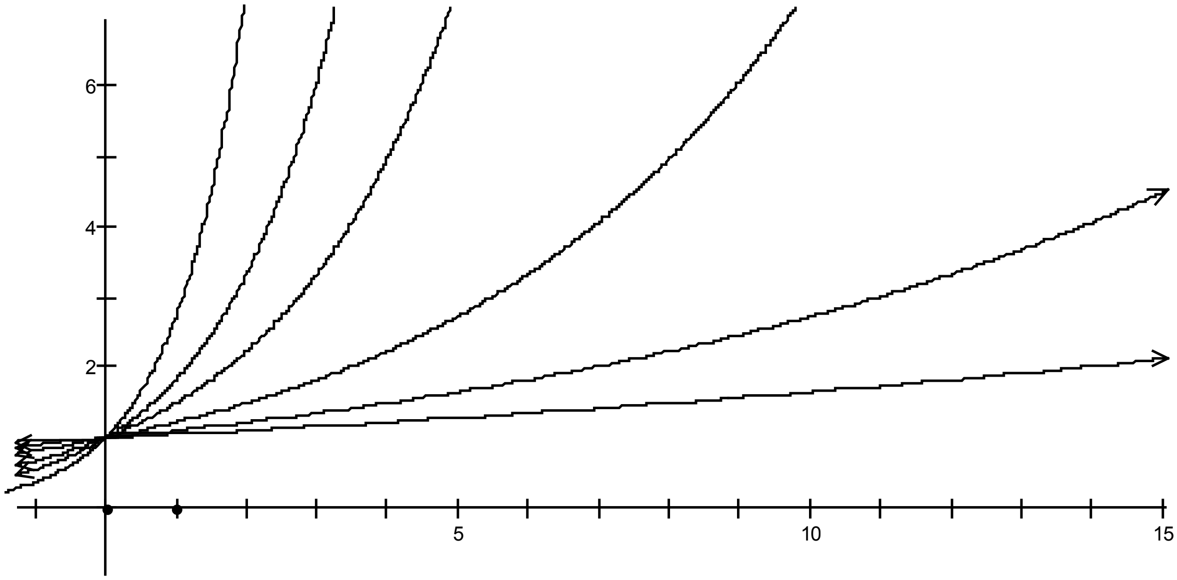
We check here by noting that

$P'(t) = kCe^{kt} = kP(t)$ indicating that this is a solution.

Note that if $C \leq 0$ we are talking about a zero population, which is not very interesting, or a negative population which has no meaning at all, so assume that $C > 0$.

This of course is not a single solution, but a family of solutions.

Here are some examples for k with different values:



To narrow this down we can use some data points.

For example assume that at time time 0 the population is P_0 and at time 1 it is P_1

So we can first plug in 0 giving

$$P(0) = Ce^0 = C = P_0$$

Now we have

$$P(t) = P_0 e^{kt}$$

so plugging in 1 we get

$$P(1) = P_0 e^k = P_1$$

and therefore

$$k = \ln \frac{P_0}{P_1}$$

This differential equation is very simple and will only apply for a population with an unlimited supply of resources that it needs to grow. If we want to include the possibility that resources will run out over time we need a more complex equation, for example:

The maximum population that a limited but constant supply of resources can sustain is called the CARRYING CAPACITY.

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

Where M is "Carrying Capacity" . This equation is called the LOGISTIC DIFFERENTIAL EQUATION, but we won't look at this anymore today.

A second example, the motion of a Spring

Many springs will obey Hook's law that $F=-kx$ meaning the force returning the spring to its initial position is directly proportional to and in the opposite direction of its distance from its natural length.

Using the physics equation $F=ma$ we have $ma=-kx$ or $a=-\frac{k}{m}x$.

Note that here we mean by x the function $x(t)$

Since acceleration is the 2nd derivative of position with respect to time, this leaves us with the differential equation:

$$\frac{d^2x}{dt^2}=-\frac{k}{m}x$$

If you think for a minute about a function whose 2nd derivative is proportional to but the negative of the original function, you might come up with the sine and/or cosine.

With this idea in mind we try the function

$$x(t)=A\sin(rt)+B\cos(rt)+C$$

$$x'(t)=rA\cos(rt)-rB\sin(rt)$$

$$x''(t)=-r^2A\sin(rt)-r^2B\cos(rt)=-r^2\left[A\sin(rt)+B\cos(rt)\right]=-r^2x$$

So if we set $r=\sqrt{\frac{k}{m}}$ we have the solution $x(t)=A\sin\left(\sqrt{\frac{k}{m}}t\right)+B\cos\left(\sqrt{\frac{k}{m}}t\right)+C$

If we have the initial condition that we stretch the spring a distance D and release it at t=0, making this an INITIAL VALUE PROBLEM then we have

$$x(0)=B+C=D$$

$$x'(0)=\sqrt{\frac{k}{m}}A=0\rightarrow A=0$$

So we are left with

$$x(t) = B\cos\left(\sqrt{\frac{k}{m}}t\right) + C$$

$$x'(t) = -\sqrt{\frac{k}{m}}B\sin\left(\sqrt{\frac{k}{m}}t\right)$$

$$x''(t) = -\frac{k}{m}B\cos\left(\sqrt{\frac{k}{m}}t\right) = -\frac{k}{m}x$$

and we know that the acceleration at *t*=0 must be $-\frac{k}{m}D$ so

$$x''(0) = -\frac{k}{m}D = -\frac{k}{m}B \rightarrow B = D$$

This gives us $D + C = D$

so C=0, giving the final equation:

$$x(t) = D\cos\left(\sqrt{\frac{k}{m}}t\right)$$

General Differential Equation

The first differential equation we looked at only had a first derivative so it is a differential equation of the first ORDER.

The second differential equation had a second derivative so it is second order.

Both only have the function variable to the first degree so they are both first degree or LINEAR differential equations.

For the physics majors in the audience, it is useful to know that most differential equations found in nature first or second order linear equations. A notable exception is found in Einstein's theory of General Relativity or Gravity in which you find non-linear equations.

One kind of differential equation that lends itself to a direct solution is one with separable variables.

if we have $\frac{dy}{dx} = g(x)f(y)$ where g and f only involve x and y respectively then we can perform the following hand-waved approach:

$$\frac{1}{f(y)}dy = g(x)dx$$

Note this is really notational. dy and dx are not elements that can be divided up this way, however the usefulness of this strategy shows some of the advantages of the Leibniz notation over Newtonian.

From here we proceed to $\int \frac{1}{f(y)}\,dy = \int g(x)\,dx$

Assuming we can evaluate both of these integrals, we will have an implicit solution, we may possibly be able to solve for y giving us an explicit solution.

We can verify that this hand-wave is correct by finding the derivative of both sides.

$$\frac{d}{dx}\int \frac{1}{f(y)}\,dy = \frac{d}{dx}\int g(x)\,dx$$

First we see that

$$\frac{d}{dx}\int g(x)\,dx = g(x)$$

Next using the chain rule we have

$$\frac{1}{f(y)}\frac{dy}{dx} = g(x) \rightarrow \frac{dy}{dx} g(x)f(y)$$

Putting these together we have

$$\frac{d}{dx}\int \frac{1}{f(y)}\,dy = \frac{d}{dy}\left(\int \frac{1}{f(y)}dy\right)\frac{dy}{dx} = \frac{1}{f(y)}\frac{dy}{dx}$$

Example:

$$\frac{dy}{dx}=\frac{x^2}{y^2}\rightarrow\int y^2dy=\int x^2dx\rightarrow\frac{y^3}{3}=\frac{x^3}{3}+C\rightarrow y=\sqrt[3]{x^3+3C}\rightarrow y=\sqrt[3]{x^3+D}$$

Given initial conditions $y(0)=1$ we have $\sqrt[3]{D}=1\rightarrow D=1$ for a solution of $y=\sqrt[3]{x^3+1}$

Example: with an implicit solution

$$\frac{dy}{dx}=\frac{6x^2}{2y+\cos y}\rightarrow\int 2y+\cos y\ dy=\int 6x^2dx\rightarrow y^2+\sin y=2x^3+C$$

Here it is impossible to solve for *y,* however any function that fulfills this equation will be a solution to the differential equation.

Example:

$$\frac{dy}{dx}=x^2y$$

$$\int\frac{dy}{y}=\int x^2dx\rightarrow\ln|y|=\frac{x^3}{3}+C$$

We raise both sides to the power if *e*:

$$e^{\ln|y|}=e^Ce^{x^3/3}\rightarrow|y|=De^{x^3/3}\rightarrow y=\pm De^{x^3/3}$$

Note that $D=0\rightarrow y=0$ is also a solution, so we have

$$y=Ae^{x^3/3}\quad A\in\mathbb{R}$$

Distribute Handout