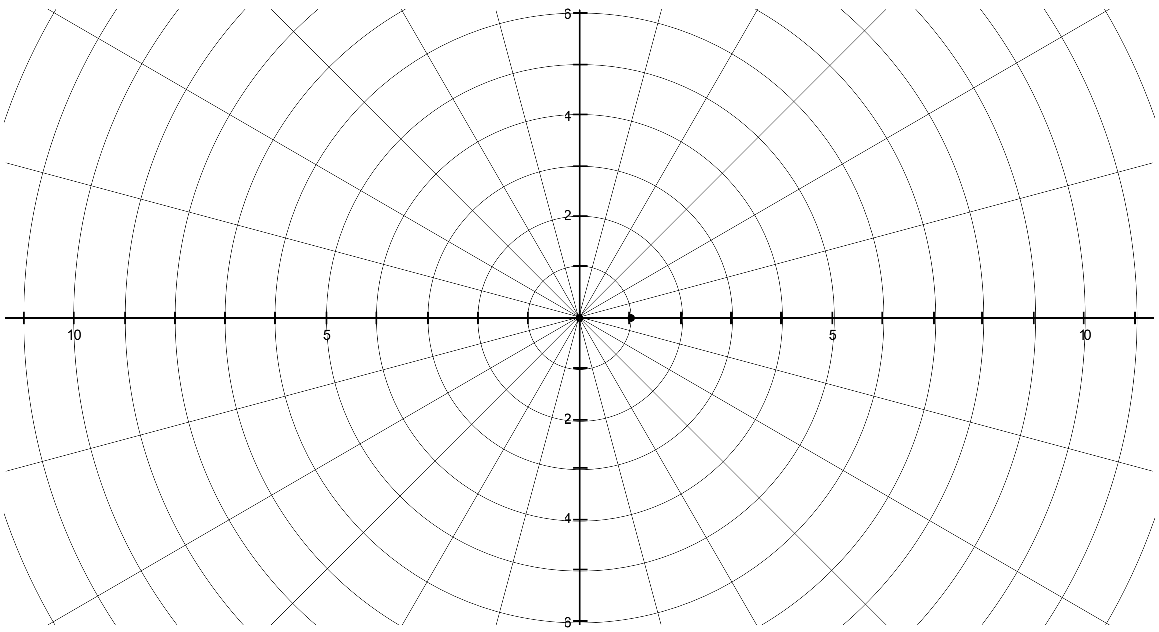
Lesson Plan 10 - Polar Coordinates 10.3

1) Take attendance
2) Homework questions?

Polar Coordinates.

What are Polar Coordinates?

$r \quad and\ \theta$



Each point has two coordinates (r,θ) instead of (x,y)

Polar coordinates are not unique, for example

$(0,1)=(0,2)$

$(1,0)=(1,2\pi)$

$(1,0)=(-1,\pi)$

Converting from Polar to Cartesian Coordinates:

$x = r\cos(\theta)$
$y = r\sin(\theta)$

Converting from Cartesian to Polar Coordinates:

$r = \sqrt{x^2 + y^2}$
$\theta = \tan^{-1}\left(\frac{y}{x}\right)$ if $x \neq 0$

Note that if x=0 then $\theta = \frac{\pi}{2}$ or $\theta = \frac{3\pi}{2}$

Also note that the range of $\tan^{-1}$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ so you need to look at the sign of x and y to find the right quadrant:

(+,+) - first
(-,+) - *second
(-,-) - *third
(+,-) - fourth

For the second and third quadrant you will need to add π to get the right angle.

What is a polar equation?

r=5

r=θ

$r = \cos(\theta)$

Have students graph r=1+cos(θ) on their calculators. This is what is known as a "cartoid" because it looks like a heart.

Have students graph r=2cos(θ) on their calculators. This should be a circle.

Have students graph r=cos(2θ) on their calculators. This should be a clover leaf.

Symmetry in Polar coordinates.

A) If an equation is unchanged by substituting $-\theta$ for θ then it is symmetric about the line θ=0 or the x axis.

B) If an equation is unchanged by substituting -r for r then it is symmetric about the origin.

C) If the equation is unchanged by replacing θ with $\pi-\theta$, then it is symmetric about the line $\theta=\frac{\pi}{2}$

Area of a sector

The area of a circle is πr^2.

So the area per radian of θ is $\frac{\pi r^2}{2\pi}=\frac{1}{2}r^2$

So if we have a curve in polar coordinates where r is a function of θ, $r=f(\theta)$ then the area contained between the limits of the curve and the curve is given by

$$A=\int_a^b \frac{1}{2}\left[f(\theta)\right]^2 d\theta$$

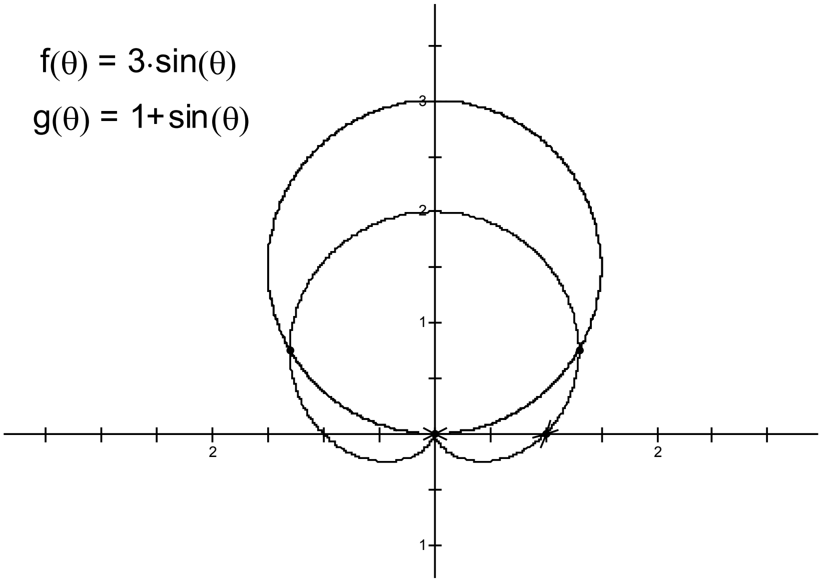
Example 1: Find the area enclosed by one loop of the four leaved rose $r=\cos(2\theta)$.

The area is swept out over the interval $\theta\in\left[-\frac{\pi}{4},\frac{\pi}{4}\right]$ so the area is

$$A=\int_{-\pi/4}^{\pi/4}\frac{1}{2}\left[\cos(2\theta)\right]^2 d\theta=\int_{-\pi/4}^{\pi/4}\frac{1}{4}\left(1+\cos(4\theta)\right)d\theta=\frac{1}{4}\left[\theta+\frac{\sin(4\theta)}{4}\right]_{-\pi/4}^{\pi/4}=$$

$$\frac{1}{4}\left[\frac{\pi}{4}+\frac{\sin(\pi)}{4}-\left(-\frac{\pi}{4}+\frac{\sin(-\pi)}{4}\right)\right]=\frac{\pi}{8}$$

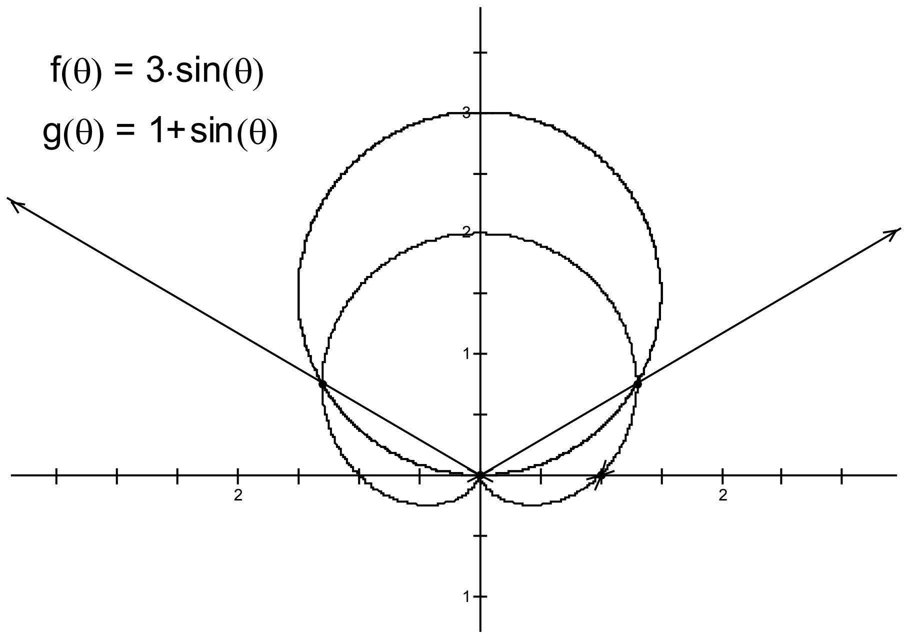
Example 2: Find the area of thre region that lies inside the circle $r = 3$ and outside the cartioid $r = 1 + \sin\theta$.



It's interesting to note that both curves go through the origin, but $f(0) = 0$ but $g\left(\frac{3\pi}{2}\right) = 0$. Since $(0,0) = \left(0, \frac{3\pi}{2}\right)$ they intersect at the origin.

Where else do they intersect?

$3\sin(\theta)=1+\sin(\theta)$ at $\sin(\theta)=\frac{1}{2}$. Looking at the diagram one can see that these are the angles $\theta=\frac{\pi}{6}$ and $\theta=\frac{5\pi}{6.}$



So $A=\frac{1}{2}\int_{\pi/6}^{5\pi/6}\left(3\sin(\theta)\right)^2 d\theta-\frac{1}{2}\left(1+\sin\theta\right)^2 d\theta$

Arc Length in Polar Coordinates

To find the length of a polar curve $r = f(\theta)$ for $a \le \theta \le b$ we use the equations

$$x = r\cos\theta = f(\theta)\cos(\theta)$$
$$y = r\sin\theta = f(\theta)\sin(\theta)$$

with θ as a parameter.

Using the arc length formula we get

$$L = \int_a^b \sqrt{\left(\frac{d}{d\theta} f(\theta)\cos(\theta)\right)^2 + \left(\frac{d}{d\theta} f(\theta)\sin(\theta)\right)^2}\, d\theta =$$

$$L = \int_a^b \sqrt{\left(\cos(\theta) f'(\theta) + -\sin(\theta) f(\theta)\right)^2 + \left(\sin(\theta) f'(\theta) + \cos(\theta) f(\theta)\right)^2}\, d\theta =$$

$$L = \int_a^b \sqrt{\begin{array}{l} \left(\cos(\theta) f'(\theta)\right)^2 - 2\cos(\theta)\sin(\theta) f(\theta) f'(\theta) + \left(\sin(\theta) f(\theta)\right)^2 + \\ \left(\sin(\theta) f'(\theta)\right)^2 + 2\cos(\theta)\sin(\theta) f(\theta) f'(\theta) + \left(\cos(\theta) f(\theta)\right)^2 \end{array}}\, d\theta =$$

$$L = \int_a^b \sqrt{\left(f'(\theta)\right)^2 + \left(f(\theta)\right)^2}\, d\theta$$

Example 4:
Find the length of the cardioid with $r = 1 + \sin\theta$.

Note this cardioid is generated by $\theta \in [0, 2\pi]$

$$f(\theta) = 1 + \sin(\theta)$$
$$f'(\theta) = \cos(\theta)$$
$$L = \sqrt{\int_0^{2\pi} \left(1+\sin(\theta)\right)^2 + \cos^2(\theta) d\theta} = \int_0^{2\pi} \sqrt{1 + 2\sin(\theta) + \sin^2(\theta) + \cos^2(\theta)}\, d\theta =$$

$$\int_0^{2\pi} \sqrt{2 + 2\sin(\theta)}\, d\theta = 8 \text{ (By Calculator)}$$