M1B/Schoenbrun Section 6.2 Regions Between Curves 1)

1) Find the area between $-x^2+5$ and $-x+3$

First find the points of intersection by equating the two

$-x^2+5=-x+3 \rightarrow x^2-x-2=0 \rightarrow (x-2)(x+1)=0 \rightarrow x=-1,2$

The parabola is greater than the line on the interval so

$$A=\int_{-1}^{2} -x^2+5-(-x+3)\,dx=\int_{-1}^{2} -x^2+x-2\,dx=\left[\frac{-x^3}{3}+\frac{x^2}{2}-2x\right]_{-1}^{2}=$$

$$\left[\frac{-8}{3}+\frac{4}{2}-4-\left(\frac{1}{3}+\frac{1}{2}-2\right)\right]=\frac{9}{2}$$

2) Find the area in the 1st quadrant between $y=2x$ and $y=x\sqrt{3x^2+1}$

Find the points of intersection by equating the two functions:

$2x=x\sqrt{3x^2+1} \rightarrow 2x-x\sqrt{3x^2+1}=0 \rightarrow x\left(2-\sqrt{3x^2+1}\right)=0, x=0$

$2-\sqrt{3x^2+1}=0 \rightarrow 4=3x^2+1 \rightarrow x^2=\pm 1$

So the interval is [0,1]

$$A=\int_0^1 2x-x\sqrt{3x^2+1}\,dx=\int_0^1 2x\,dx-\int_0^1 x\sqrt{3x^2+1}\,dx$$

The first integral is easily found to be 1. To solve the second use substitution.

$u=3x^2+1$

$\frac{du}{6}=x\,dx$

$$\int_0^1 x\sqrt{3x^2+1}\,dx=\int_1^4 \frac{\sqrt{u}}{6}\,du=\left[\frac{u^{3/2}}{9}\right]_1^4=\frac{8}{9}-\frac{1}{9}=\frac{7}{9}$$

So $A=1-\frac{7}{9}=\frac{2}{9}$

3) Find the area bounded by $y=\sqrt{\frac{x}{2}+1}$, $y=\sqrt{1-x}$, and $y=0$.

Integrate along the y axis.

Find the points of intersection by equating the two functions:

$$\sqrt{\frac{x}{2}+1}=\sqrt{1-x}\rightarrow\frac{x}{2}+1=1-x\rightarrow x=0$$

At x=0, y=1, so integrating along y we get:

$$\int_0^1\sqrt{\frac{x}{2}+1}-\sqrt{1-x}\,dx=\int_0^1\sqrt{\frac{x}{2}+1}\,dx-\int_0^1\sqrt{1-x}\,dx$$

To solve the first integral substitute

$$u=\frac{x}{2}+1$$

$$2du=dx$$

$$\int_0^1\sqrt{\frac{x}{2}+1}\,dx=\int_1^{3/2}2\sqrt{u}\,du=\frac{4}{3}\left[u^{3/2}\right]_1^{3/2}=\frac{4}{3}\cdot\frac{3\sqrt{3}}{2\sqrt{2}}-\frac{4}{3}=\sqrt{6}-\frac{4}{3}$$

To solve the second integral

$$u=1-x$$

$$du=-dx$$

$$\int_0^1\sqrt{1-x}\,dx=-\int_1^0\sqrt{u}\,du=\left[\frac{2u^{3/2}}{3}\right]_0^1=\frac{2}{3}$$

So $A=\sqrt{6}-\frac{4}{3}-\frac{2}{3}=\sqrt{6}-2$

4) Plot the area between $y=x^2$ and $y=x$ without breaking up the integral, but instead using geometry.

By inspection, the curves intersect at 0 and 1. The area under y=x can be found using the triangle area formula $A=\frac{1}{2}bh=\frac{1}{2}\cdot1\cdot1=\frac{1}{2}$

So we can calculate the area between the curves as:

$$A=\frac{1}{2}-\int_0^1 x^2\,dx=\frac{1}{2}-\left[\frac{x^3}{3}\right]_0^1=\frac{1}{2}-\frac{1}{3}=\frac{1}{6}$$

5) Find the area between $y = \frac{x}{\sqrt{x^2+1}}$ and $x^4 - x$

You should approximate the intersection of the curves with your calculator.

You can see by inspection that one of the intersection points is at [0,0]. Using your calculator you should find the 2nd point at 1.1170213.

At this point you can either find the anti derivatives, $\sqrt{x^2+1}$ and $\frac{x^5}{5} - \frac{x^2}{2}$ or just enter

$$\int_0^{1.1170213} \frac{x}{\sqrt{x^2+1}} - \left(x^4 - x\right) dx$$

finding the approximate answer .77530747

6) Find the area between x^n and x^{n+1} where $n \in \mathbb{N} = \{1,2,3,\ldots\}$

Equating the two curves we find $x^n = x^{n+1} \to x^n - x^{n+1} = 0 \to x^n\left(1-x\right) = 0 \to x = 1,0$

So $A = \int_0^1 x^n - x^{n+1}\,dx = \left[\frac{x^{n+1}}{n+1} - \frac{x^{n+2}}{n+2}\right]_0^1 = \frac{1}{n+1} - \frac{1}{n+2} = \frac{1}{n^2+3n+2}$