Section 6.2 Page. 416 (8, 12, 16, 24)

8) Determine the area of the shaded region in the following figures:

$y = 4\cos^2 x$ and $y = \dfrac{\sec^2 x}{4}$

To find the points of intersection, set the two functions equal to each other.

$$4\cos^2 x = \frac{\sec^2 x}{4} \to \cos^4 x = \frac{1}{16} \to \cos x = \pm\frac{1}{2} \to x = \pm\frac{\pi}{3}$$

so

$$A = \int_{-\pi/3}^{\pi/3} 4\cos^2 x - \frac{\sec^2 x}{4}\, dx$$

To find the anti-derivative of $\cos^2 x$ note the trigonometric identity:

$$\cos 2x = 2\cos^2 x - 1 \to \cos^2 x = \frac{\cos 2x + 1}{2}$$

also recall that $\dfrac{d}{dx}\tan x = \sec^2 x$

$$\int_{-\pi/3}^{\pi/3} 4\cos^2 x - \frac{\sec^2 x}{4}\, dx = \int_{-\pi/3}^{\pi/3} 4\left(\frac{\cos 2x + 1}{2}\right) - \frac{1}{4}\sec^2 x\, dx = \int_{-\pi/3}^{\pi/3} 2\cos 2x + 2 - \frac{1}{4}\sec^2 x\, dx =$$

$$\left[\sin 2x + 2x - \frac{1}{4}\tan x\right]_{-\pi/3}^{\pi/3} = \left(\frac{\sqrt{3}}{2} + \frac{2\pi}{3} - \frac{\sqrt{3}}{4}\right) - \left(\frac{-\sqrt{3}}{2} + \frac{-2\pi}{3} - \frac{-\sqrt{3}}{4}\right) = \frac{4\pi}{3} + \frac{\sqrt{3}}{2}$$

12) Sketch the region and find its area. The region bounded by $y = 2x$ and $y = x^2 + 3x + 16$

To find the points of intersection, set the functions equal:

$$2x = x^2 + 3x - 6 \to x^2 + x - 6 = 0 \to (x-2)(x+3) = 0 \to x = 2, -3$$

$$A = \int_{-3}^{2} 2x - \left(x^2 + 3x - 6\right) dx = \int_{-3}^{2} -x^2 - x + 6\, dx = -\frac{x^3}{3} - \frac{x^2}{2} + 6x\Big|_{-3}^{2} =$$

$$\left(-\frac{8}{3} - 2 + 12\right) - \left(9 - \frac{9}{2} - 18\right) = \frac{125}{6}$$

16) Find the area between $y = \sin x$ and $y = \sin 2x$ on the interval $[0, \pi]$.

To find the points of intersection, set the functions equal:

$\sin x = \sin 2x$

Using the trigonometric identity:

$\sin 2x = 2\sin x \cos x$

$\sin x = 2\sin x \cos x \rightarrow 2\sin x \cos x - \sin x = 0 \rightarrow \sin x(2\cos x - 1) = 0 \rightarrow x = 0, \frac{\pi}{3}, \pi$

$$A = \int_0^{\pi} |\sin x - \sin 2x|\, dx = \int_0^{\pi/3} \sin 2x - \sin x\, dx + \int_{\pi/3}^{\pi} \sin x - \sin 2x\, dx =$$

$$\frac{-\cos 2x}{2} - {}^{-}\cos x \Big|_0^{\pi/3} + -\cos x - \frac{-\cos 2x}{2} \Big|_{\pi/3}^{\pi} =$$

$$\left[\frac{-\cos 2x}{2} + \cos x \Big|_0^{\pi/3} \right] + \left[-\cos x + \frac{\cos 2x}{2} \Big|_{\pi/3}^{\pi} \right] =$$

$$\left[\left(\frac{1}{4} + \frac{1}{2} \right) - \left(\frac{-1}{2} + 1 \right) \right] + \left[\left(1 + \frac{1}{2} \right) - \left(\frac{-1}{2} + \frac{-1/2}{2} \right) \right] = \frac{1}{4} + \frac{9}{4} = \frac{5}{2}$$

24) Find the area of the region bounded by $x = \cos y$ and $x = -\sin 2y$

To find the points of intersection, set the functions equal:

$\cos y = -\sin 2y \rightarrow \cos y = -2\sin y \cos y \rightarrow 2\sin y \cos y + \cos y = 0 \rightarrow$

$\cos y(2\sin y + 1) = 0 \rightarrow \cos y = 0, \sin y = -\frac{1}{2} \rightarrow y = \frac{\pi}{2}, -\frac{\pi}{6}$

$$A = \int_{-\pi/6}^{\pi/2} \cos y + \sin 2y\, dy = \left[\sin y - \frac{\cos 2y}{2} \right]_{-\pi/6}^{\pi/2} =$$

$$\left(1 + \frac{1}{2} \right) - \left(-\frac{1}{2} - \frac{1}{3} \right) = \frac{9}{4}$$