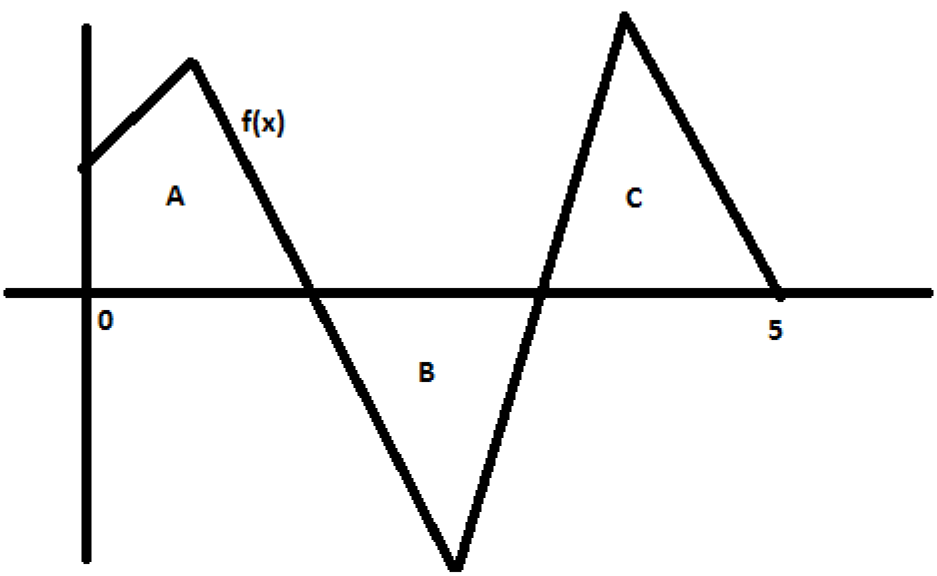
Foothill College Math 1B - Final Exam Mitchell Schoenbrun

1) Given the diagram of $f\left(x\right)$ below with area A = 5, area B = 6 and area C = 4, evaluate the following integrals

$$\int_0^5 f\left(x\right)\,dx = \underline{5+(-6)+4=3}$$

$$\int_0^5 \left|f\left(x\right)\right|\,dx = \underline{|5|+|(-6)|+|4|=15}$$



2) Evaluate the following Integral EXACTLY

$$\int_2^3 \frac{2x^2+3}{x^2(x-1)}\,dx = \int_2^3 \frac{A}{x^2}+\frac{B}{x}+\frac{C}{x-1}\,dx$$

$$Ax - A + Bx^2 - Bx + Cx^2 = 2x^2 + 3$$

$$A = -3$$

$$B = -3$$

$$C = 5$$

$$\int_2^3 \frac{2x^2+3}{x^2(x-1)} = \int_2^3 \frac{-3}{x^2}+\frac{-3}{x}+\frac{5}{x-1}\,dx =$$

$$\left[\frac{3}{x} - 3\ln|x| + 5\ln|x-1|\right]_2^3 = -\frac{1}{2} - 3\ln 3 + 8\ln 2$$

3) Evaluate the following indefinite integral

$$\int \cos^3 x\,dx = \int \cos x\left(1-\sin^2 x\right)dx = \int \cos x - \cos x \sin^2 x\,dx =$$

$$\sin x - \frac{\sin^3 x}{3} + C$$

Note: Other equivalent solutions are possible

4) Find the Average value EXACTLY of the function $f\left(x\right)=x^2\ln\left(x\right)$ on the interval $\left[1,3\right]$

$$Avg=\frac{\int_1^3 x^2\ln x\,dx}{3-1}$$

$$f=\ln x \quad g'=x^2$$

$$f'=\frac{1}{x} \quad g=\frac{x^3}{3}$$

$$\int_1^3 x^2\ln x\,dx=\left[\frac{x^3}{3}\ln x-\int\frac{x^2}{3}\,dx\right]_1^3=9\ln 3-\frac{26}{9}$$

$$Avg=\frac{9\ln 3}{2}-\frac{13}{9}$$

5) The cross section of a solid is a triangle with base x and height x^2.
What is the EXACT volume of this solid on the x interval $\left[1,5\right]$

$$Area=\frac{1}{2}x\cdot x^2=\frac{x^3}{2}$$

$$Volume=\int_1^5\frac{x^3}{2}dx=\frac{x^4}{8}\Big|_1^5=\frac{625-1}{8}=78$$

6) Find the EXACT length of the curve described by the function

$f(x)=\frac{\sqrt{x}}{3}(x-3)$ on the interval $[0,4]$

$$f'(x)=\frac{d}{dx}\left(\frac{x^{3/2}}{3}-x^{1/2}\right)=\frac{1}{2}\left(x^{1/2}-x^{-1/2}\right)$$

$$Len=\int_0^4\sqrt{1+\left(f'(x)\right)^2}\,dx=\int_0^4\sqrt{1+\left(\frac{1}{2}\left(x^{1/2}-x^{-1/2}\right)\right)^2}\,dx=$$

$$\int_0^4\sqrt{\frac{4}{4}+\frac{1}{4}\left(x-2+x^{-1}\right)}dx=\int_0^4\sqrt{\frac{1}{4}\left(x+2+x^{-1}\right)}dx=$$

$$\int_0^4\sqrt{\left(\frac{x^{1/2}+x^{-1/2}}{2}\right)^2}\,dx=\int_0^4\frac{x^{1/2}+x^{-1/2}}{2}\,dx=$$

$$\frac{1}{2}\left[\frac{2x^{3/2}}{3}-\frac{x^{1/2}}{2}\right]_0^4=\frac{14}{3}$$

7) Find $\frac{d}{dx}f(x)$ given that $f(x)=\int_0^{x^2}\left(g(u)\right)^2du$

Applying the fundamental theorem of Calculus and the chain rule we find that

$$\frac{d}{dx}=\int_0^{x^2}\left(g(u)\right)^2du=\left(g\left(x^2\right)\right)^2\cdot\frac{d}{dx}x^2=2x\left(g\left(x^2\right)\right)^2$$

8) Use your calculator to find the approximate area enclosed by the functions

$$f(x) = 2\cos x$$
$$g(x) = 2\sec x - 1$$

between their intersection on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

1 Method: Graph $f(x) - g(x) = 2\cos x - 2\sec x - 1$
and use the Calc: Zero function.

Note that both functions are even so their difference is even.

zero = $\pm.67488885$

$$Area = \int_{-.6748885}^{.67488885} 2\cos x - (2\sec x - 1)\,dx = 2\int_{0}^{.67488885} 2\cos x - (2\sec x - 1)\,dx$$

Now use the Calc: Integrate function to find the area = .9175891227

9) Evaluate the following two improper integrals EXACTLY where possible.

$$\int_0^\infty \frac{1}{e^x}\,dx = \lim_{a\to\infty}\int_0^a \frac{1}{e^x}\,dx = \lim_{a\to\infty}\left[-\frac{1}{e^x}\right]_0^a = \lim_{a\to\infty}\left[-\frac{1}{e^x} - {}^{-}1\right] = 0 + 1 = 1$$

$$\int_0^\infty \frac{1}{\sqrt{x}}\,dx = \lim_{a\to\infty}\int_0^a x^{-1/2}\,dx = \lim_{a\to\infty}\left[\frac{x^{1/2}}{2}\right]_0^a = \lim_{a\to\infty}\frac{\sqrt{a}}{2} \to \infty$$

So this integral is DIVERGENT!

10) Find a solution to the differential equation $\frac{dy}{dx} = \frac{3x^2}{y}$ with the initial condition that $y(0) = 1$

Separating Variables we get

$$\int y\,dy = \int 3x^2 dx \rightarrow \frac{y^2}{2} = x^3 + C$$

Plugging in the initial condition we find

$C = \frac{1}{2}$ so the solution is $y^2 = 2x^3 + 1$ or $y = \sqrt{2x^3 + 1}$

Note the 1 must be under the square root sign.

Extra Credit)
Solve Problem 8) EXACTLY!

Setting $2\cos x = \dfrac{2}{\cos x} - 1$ and solving we find $\cos x = \dfrac{-1 \pm \sqrt{17}}{4}$.

The root $\cos x = \dfrac{-1-\sqrt{17}}{4}$ would have a cos < -1 so it is extraneous and we can now see that $x = \cos^{-1}\left(\dfrac{-1+\sqrt{17}}{4}\right)$

The intersection points are therefore

$\cos^{-1}\left(\dfrac{-1+\sqrt{17}}{4}\right)$ and $-\cos^{-1}\left(\dfrac{-1+\sqrt{17}}{4}\right) = \cos^{-1}\left(\dfrac{\sqrt{17}-1}{4}\right)$

So the Area is

$$\int_{\cos^{-1}\left(\frac{1-\sqrt{17}}{4}\right)}^{\cos^{-1}\left(\frac{\sqrt{17}-1}{4}\right)} 2\cos x - (2\sec x - 1)\,dx = 2\int_{0}^{\cos^{-1}\left(\frac{\sqrt{17}-1}{4}\right)} 2\cos x - 2\sec x + 1\,dx =$$

$$2\Big[2\sin x - 2\ln\left|\tan x + \sec x\right| + x\Big]_{0}^{\cos^{-1}\left(\frac{\sqrt{17}-1}{4}\right)}$$

This can be simplified more but the above answer or equivalent was sufficient.